Topology in Chaos

A tangled tale about knot, link, template, and strange attractor

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Topology, and Dynamical Systems Theory

Topology

Topology is the mathematical study of shapes and spaces, focuses on properties that are preserved under continuous deformations including stretching and bending, but not tearing or gluing.
One interesting branch is the knot theory

Dynamical Systems Theory

Dynamical Systems Theory deals with the long-term qualitative behavior of dynamical systems, and studies of the solutions to the equations of motion of systems. Much of current research is focused on the study of chaotic systems.

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This talk will show you the interesting and crucial connection between these two important areas of mathematics...

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2 Knot Theory

3 Nonlinear Dynamics

4 Topology in Chaos

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What is a knot

Figure: Knot in daily life. It is said that before the invention of writing characters, the ancient people kept records by tying knots.

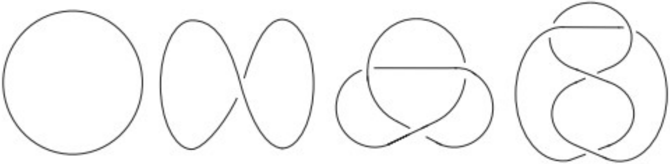
Figure: Beautiful Chinese knot

Mathematical knot

While inspired by knots which appear in daily life in shoelaces and ropes, a mathematician's knot differs in that the ends are joined together so that it cannot be undone.

Mathematical definition of knot

A *knot* is an embedding of a *circle* in 3-dimensional Euclidean space $e : S^1 \to \mathbb{R}^3$, that is $e(S^1) = K \subset \mathbb{R}^3$. Two knots $K, K^\star$ have the same *type* if there is a diffeomorphism of pairs $(K, \mathbb{R}^3) \to (K^\star, \mathbb{R}^3)$



a: the unknot b: the trivial knot c: trefoil knot d: Figure-8 knot

Knot equivalence

Two mathematical knots are equivalent if one can be transformed into the other via a deformation of R^3 upon itself. Roughly, this means not allow cutting the string.

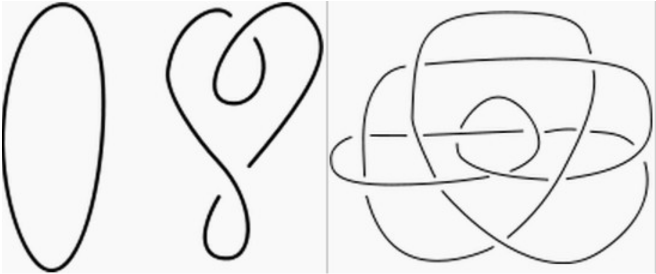
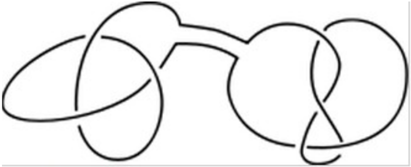


Figure: On the left, the *unknot*, and a knot equivalent to it. It can be quite difficult to determine whether complex knots, such as the one on the right, are equivalent to the unknot.

Knot sum

- Two knots can be added by cutting both knots and joining the pairs of their ends. This operation is denoted by $\#$ and the result is called the ”sum” of the two knots



Adding two knots

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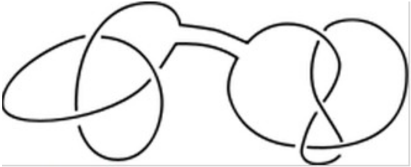

Adding two knots

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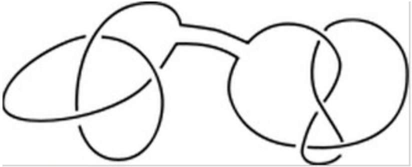

Adding two knots

- Note also that if O is a 'unknot', and a knot K, then $K\#O$ just gives us a new picture of K.
- Caution: there do not exist knots K, K', both different from the unknot, with $K\#K'$ the unknot O. That is, knots form an addition semi-group, using the operation $\#$.

Prime knot

- A knot is **prime** if it is non-trivial and cannot be written as the knot sum of two non-trivial knots.

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Schubert's theorem

Every knot can be uniquely expressed as a connected sum of prime knots.

Prime knots are the building blocks

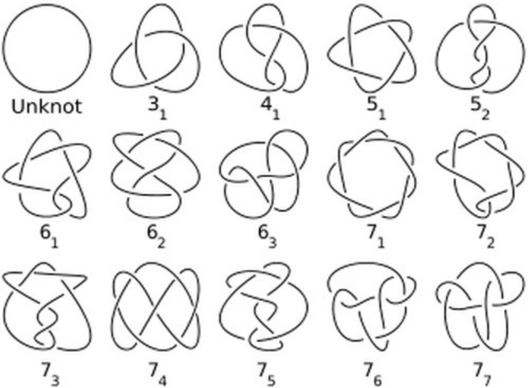


Figure: Prime knots are the building blocks of all knots. A table of prime knots up to seven crossings.

Mathematical definition of Link

Link

A *link* L is the image under an embedding of $N \geq 1$ disjoint circles in $\mathbb{R}^3$. If $N = 1$ it's a *knot*.
Two links $L, L^\star$ are the same *type* if there is a diffeomorphism of pairs $(L, \mathbb{R}^3) \to (L^\star, \mathbb{R}^3)$.

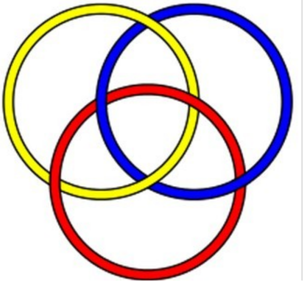
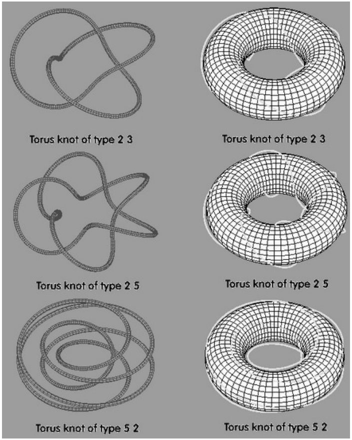


Figure: A link with three components each equivalent to the unknot

Torus knots and links

- If we restrict our attention to special classes of knots or links, then the search for invariants can sometimes be very much simpler than in the general case.
- A **torus knot** is a special kind of knot that lies on the surface of an unknotted torus in R^3



Torus knots and links

Torus knots are a well-understood class. They are classified by a pair of integers (p, q) (up to the indeterminacy $(p, q) \approx (q, p) \approx (-p, -q) \approx (-q, -p)$).

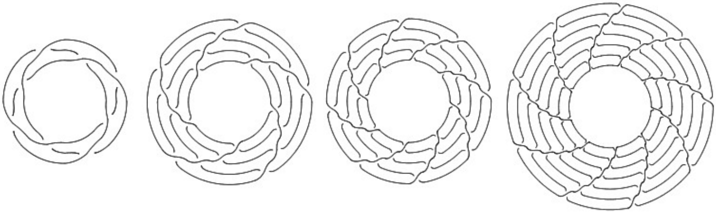


Figure: Four examples of torus knots and links. Their classifying integer pairs, reading from left to right, are $(3, 5), (5, 5), 5, 8), (7, 9)$

Summary: knots are complex...

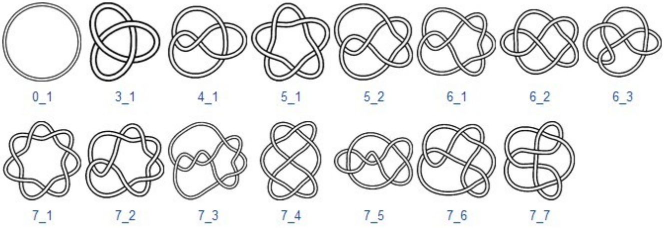


Figure: Knots with 7 or fewer crossings From `http://katlas.math.toronto.edu/wiki/The_Rolfsen_Knot_Table`

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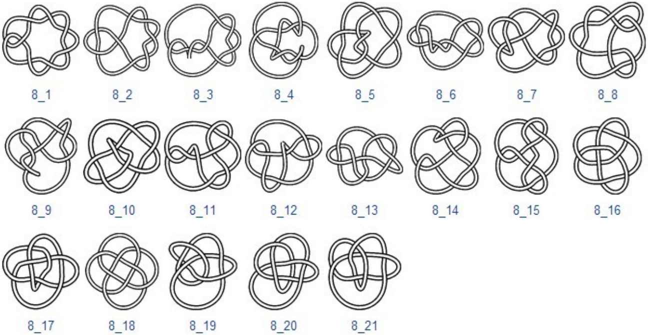

Figure: Knots with 8 crossings, the number of inequivalent knots increases...

Summary: knots are complex...

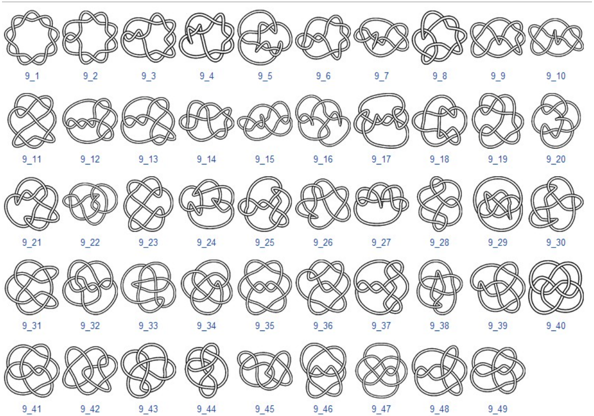

Figure: Knots with 9 crossings, the number of inequivalent knots increases dramatically... more and more complicated...

Summary: links are even more complex...

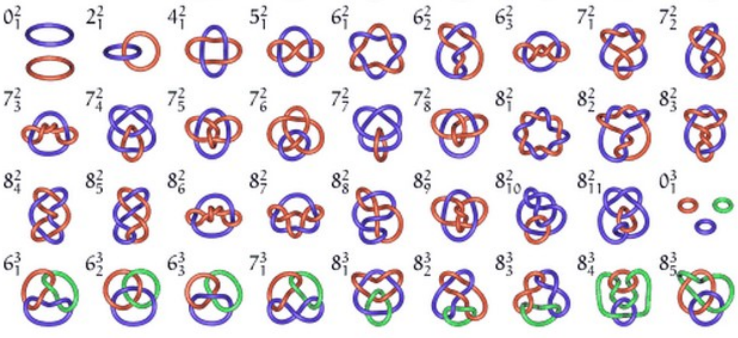

Figure: Menasco's Knot Theory Hot List. From http://www.math.buffalo.edu/ menasco/knot-theory.html

Summary: complex knots and links ...

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Summary: complex knots and links ...

- Knot theory is to understand all possible knots...too big a problem
- Prime knots are building block... still hard
- Even for same type of knots... hard to recognize
- Have to restrict our attention to special classes of knots...only very special classes are well understood today
- Links are collection of knots...

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Parametric Lorenz System

In 1963, the meteorologist Edward Lorenz was studying a highly simplified numerical model for the atmosphere, which led him to the now-classic parametric Lorenz system described by:

$$\begin{cases} \dot{x} = \sigma\,(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy\,, \end{cases} \tag{1}$$

with three real parameters $\sigma,\ r,\ b$.

Figure: Edward Lorenz

Lorenz system

- The dynamic behaviors are highly depend on the three parameters and will differ from region to region, or even point to point in the 3-dimensional parameter space.

Lorenz system

- The dynamic behaviors are highly depend on the three parameters and will differ from region to region, or even point to point in the 3-dimensional parameter space.
- When $\sigma = 10,\ r = 28,\ b = \frac{8}{3}$, this is the so-called Lorenz system and is chaotic with a strange attractor
-

$$\begin{cases} \dot{x} = 10\,(y - x) \\ \dot{y} = 28x - y - xz \\ \dot{z} = -\frac{8}{3}z + xy\,, \end{cases} \tag{2}$$

Lorenz system has a strange attractor, that is what we usually called the Lorenz attractor

Lorenz attractor

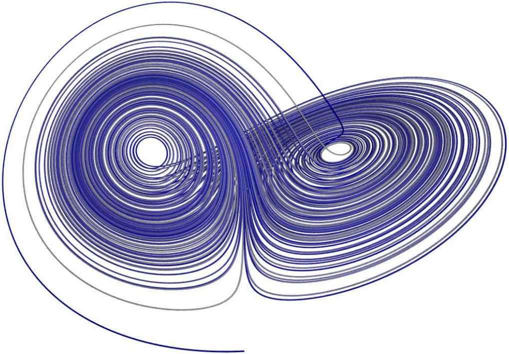

Figure: Lorenz attractor, which vividly resembles a butterfly's wings, has become an emblem of chaos; Edward N. Lorenz himself, has been marked by the history as an icon of chaos theory.

Unstable periodic orbits

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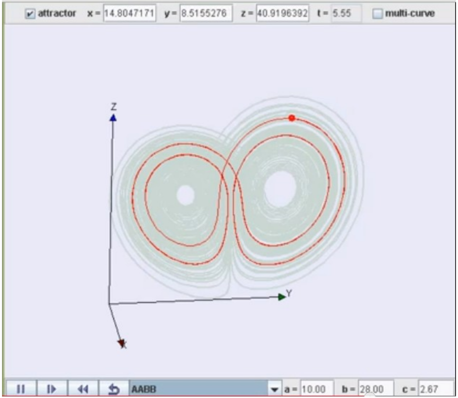
- But for some very special initial conditions it produces periodic orbits solutions. There are an infinite number of unstable periodic orbits (UPOs) embedded in the chaotic system.
- These UPOs play important roles in characterizing and analyzing the system. However, even numerically, it is not easy to detect UPOs from a continuous-time chaotic system, because they cannot be found by the forward time integration of the system.

Attractor and UPOs Animation



This video shows that Lorenz system have chaotic solutions for most initial conditions, while for some very special initial conditions it produces periodic orbits solutions.

Connection between knot topology and dynamic system

- A simple but crucial observation is that such a periodic orbit is a closed embedded curve which therefore defines a knot.
- So the study of the topology of the knots appearing in the Lorenz system could lead to better understanding of this important dynamical system.
- Any knot that could appear as a UPO in the Lorenz system is called a Lorenz knot.

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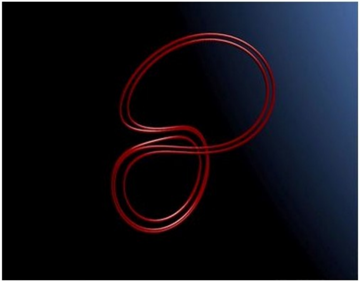
Lorenz attractor and Lorenz knot

Figure: Lorenz attractor and Lorenz knot

Some Lorenz knots are non-trivial

Some of these knots look topologically trivial, but some are not, like the red orbit which turns out to be a trefoil knot.



Some example and counter-example of Lorenz knots

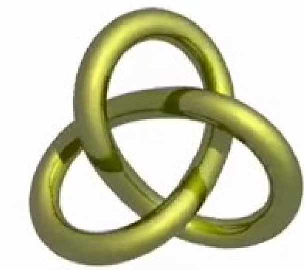


Figure: A trefoil knot

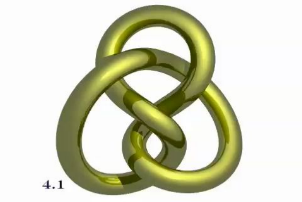


Figure: As one counter-example, the figure eight knot is not a Lorenz knot.

Lorenz knots and links are very peculiar

There are 250 (prime) knots with 10 crossings or fewer. It follows from the work of Birman and Williams that, among those 250, only the following 8 knots appear as periodic orbits of the Lorenz attractor.

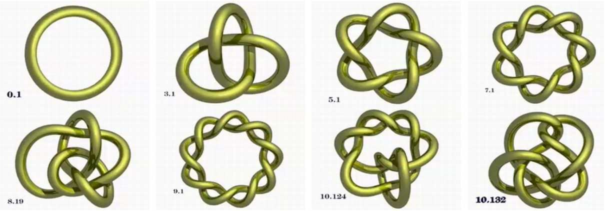


Figure: Among the 1701936 (prime) knots with 16 crossings or less, only 21 appear as Lorenz knots[1].

How to understand Lorenz attractor

Observation: Lorenz attractor appears to be a two-sheeted surface on which there is a unidirectional flow about each strip.

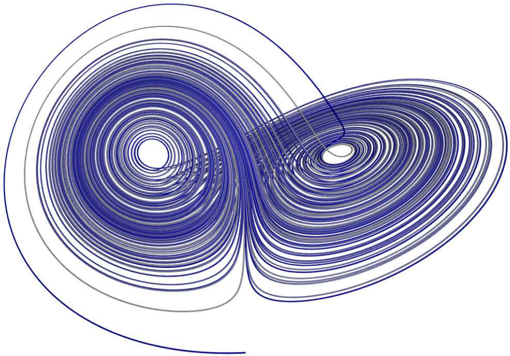


Figure: Lorenz attractor

How to understand Lorenz attractor

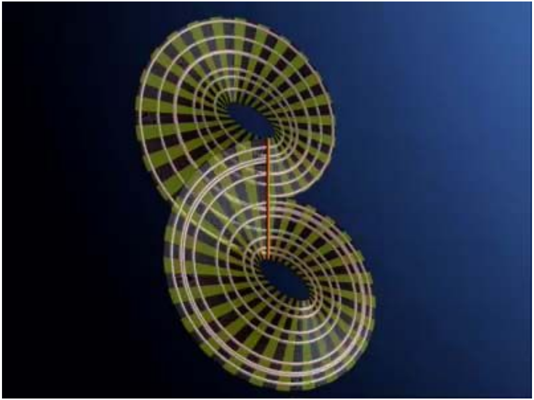


Figure: geometric Lorenz attractor. See here for animations of Lorenz Template1, 2, 3

Template

Template

A template (also known as a knot holder) is a compact branched two-manifold fitted with a smooth expansive semi-flow and built from a finite number of joining and splitting charts

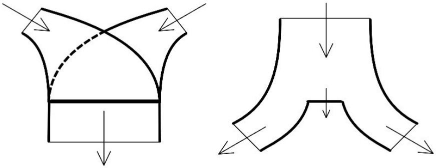
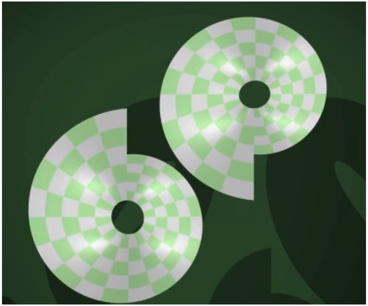
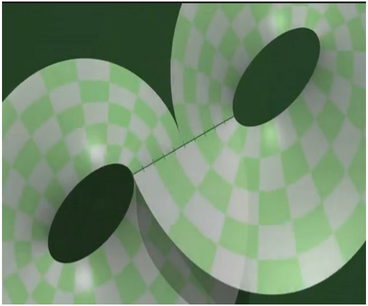


Figure: (a) joining and (b) splitting charts

Lorenz Template

How to construct this Lorenz Template?





Rössler Template

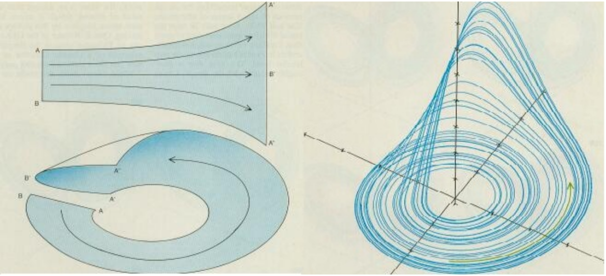
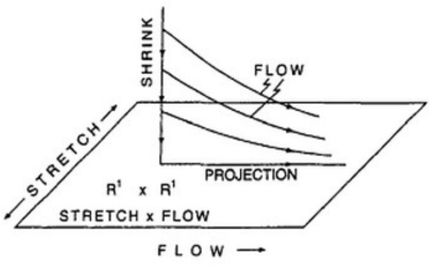


Figure: First, nearby trajectories on the object must stretch, or diverge, exponentially. Second, to keep the object compact, it must fold back onto itself

Why such simplification possible?

Assume that a dissipative dynamical system in R^3 has Lyapunov exponents $\lambda_1 > 0$ (stretching direction), $\lambda_2 = 0$ (flow direction), and $\lambda_3 < 0$ (squeezing direction), which satisfy $\lambda_1 + \lambda_2 + \lambda_3 < 0$ (dissipative condition) and produce a hyperbolic chaotic attractor.

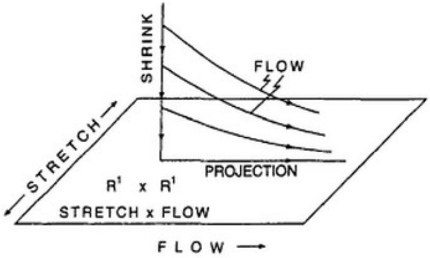


Why such simplification possible?

The identification of all points with the same future

$$x \sim y \quad \textit{if} \lim_{t\to\infty} |x(t) - y(t)| = 0$$

maps the chaotic attractor onto a branched manifold and the flow in the chaotic attractor to a semi-flow on the branched manifold.



Is the geometric Lorenz attractor arcuate?

Birman-Williams Theorem (1983) [4]

Given a flow ϕ_t on a three-dimensional manifold M^3 having a hyperbolic structure on its chain recurrent set there is a knot holder $(H, \overline{\phi}_t)$, $H \subset M^3$, such that with one or two specified exceptions the periodic orbits under ϕ_t correspond one-to-one to those under $\overline{\phi}_t$. On any finite subset of the periodic orbits the correspondence can be taken to be isotopy.

This guarantees that the organization of the unstable periodic orbits in the chaotic attractor remains unchanged under this projection.

Lorenz template and Lorenz attractor

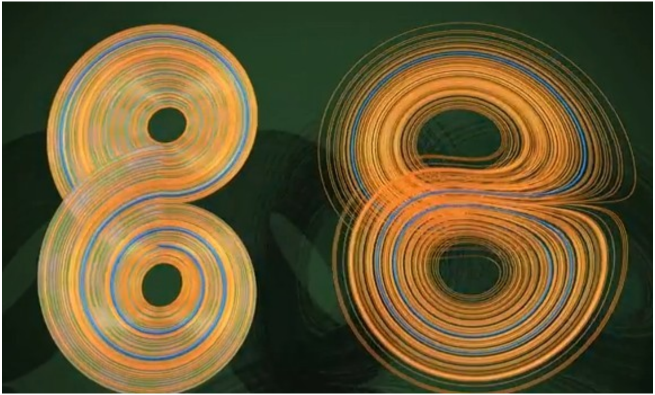


Figure: The periodic orbits in the chaotic attractor correspond in a one-to-one with the periodic orbits on the branched manifold, with only one or two exceptions.

Some conclusions about Lorenz knot and link [5, 6]

1. Lorenz knots are prime
2. All knots $\supset$ Fibered knots $\supset$ Closed + braids $\supset$ Lorenz knots
3. Every torus link is a Lorenz link
4. Non-trivial Lorenz links have a positive signature

So, Lorenz knots and links are quite special, because Lorenz template is special.
Seems each special chaotic system has its own special class of knots and links...
Rossler knot, Sprott knot, Chen knot....

An ODE whose solutions contain all knots and links

Could it possible to have an ODE system whose solutions contain all knots and links?

An ODE whose solutions contain all knots and links

Could it possible to have an ODE system whose solutions contain all knots and links? - derived from a Chua circuit:

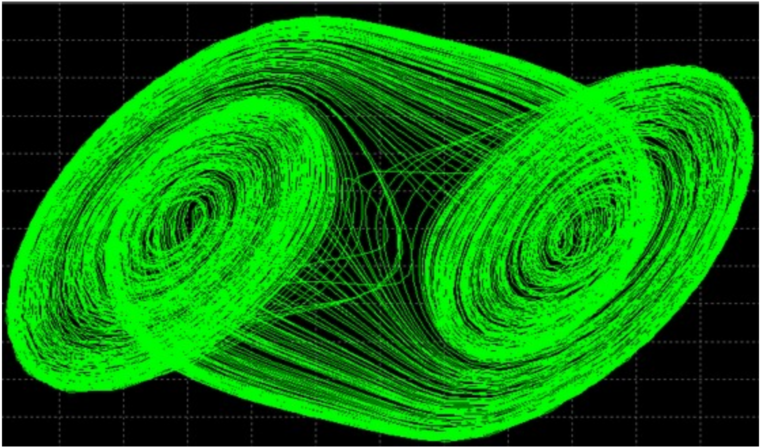
Theorem [8]

There exists an open set of parameters $\beta \in [6.5, 10.5]$ for which periodic solutions to the differential equation

$$\begin{cases} \dot{x} = 7(y - \frac{2}{7}x - \frac{3}{14}(|x+1| - |x-1|)) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y \end{cases} \tag{3}$$

contain representatives from every knot and link equivalence class.

All knots and links live here



The Universal Knot-Holder

It's because this system have a corresponding template that contains all knots

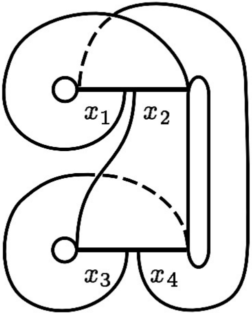


Figure: Ghrist's universal knot-holder [7].

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5 **Open Questions**

6 Summary

Chen attractor

In 1999, from an engineering feedback anti-control approach, Chen coined a new chaotic system [9], lately referred to as the Chen system by others:

$$\begin{cases} \dot{x} = 35(y - x) \\ \dot{y} = -7x + 28y - xz \\ \dot{z} = -3z + xy\,, \end{cases} \tag{4}$$

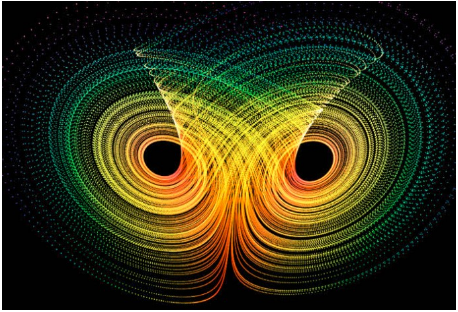

Figure: Chen Attractor

Chen knots and links?

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- What is the template of the Chen attractor?
- What are the special properties of Chen knots and links?
- What is the relation between Lorenz attractor and Chen attractor, from this topological point of view?

From Lorenz-like to Chen-like Attractors

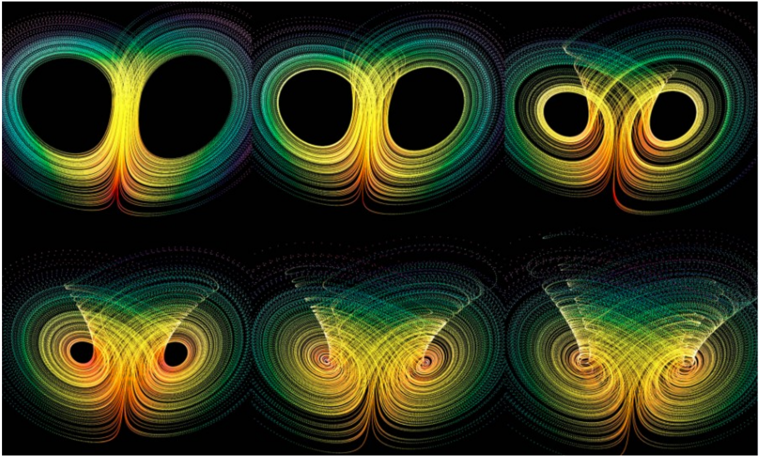

Figure: Generalized Lorenz system family

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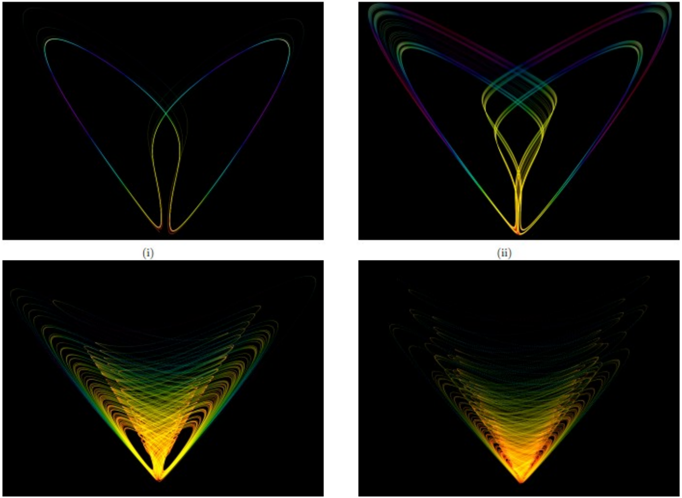
Yet More Lorenz-like to Chen-like Attractors [10]



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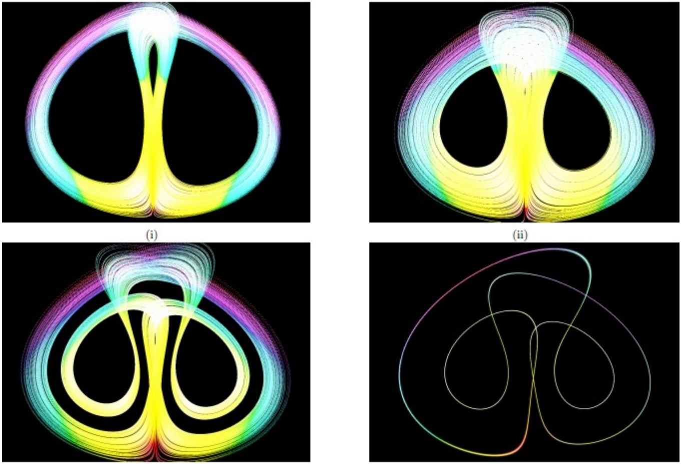
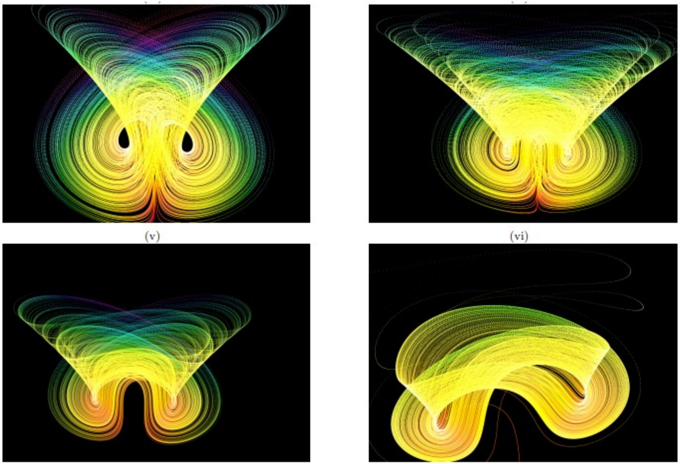
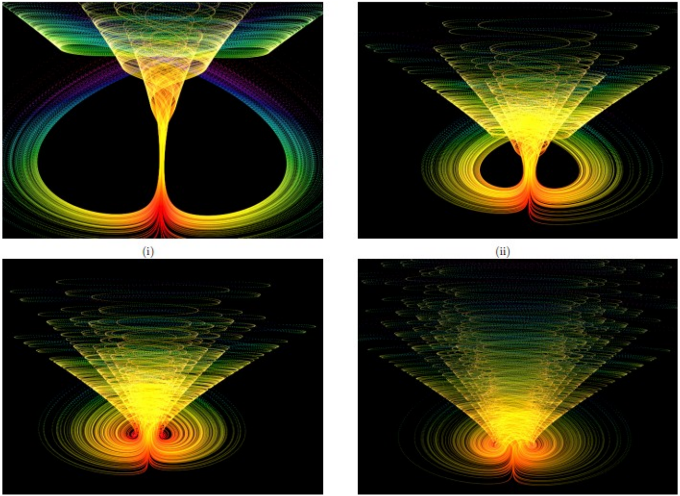


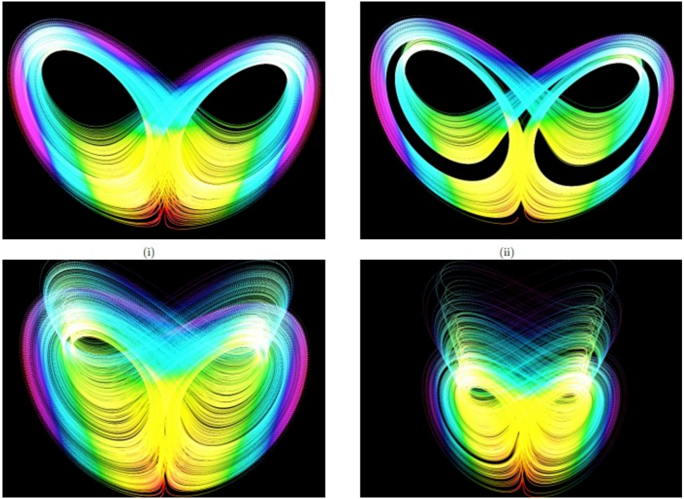
Yet More Lorenz-like to Chen-like Attractors [10]



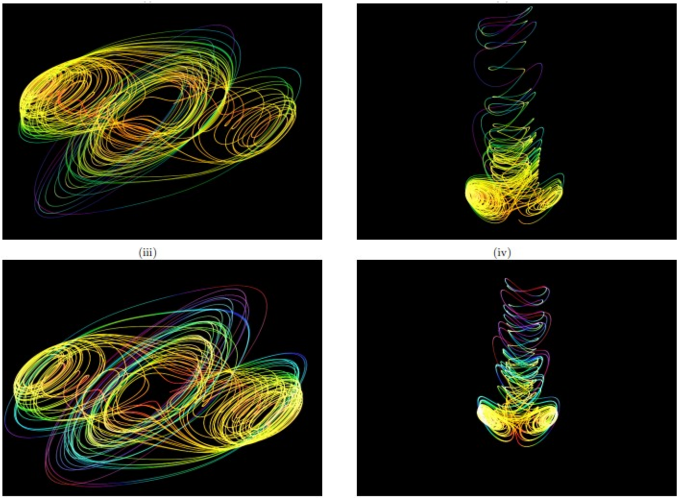
Yet More Lorenz-like to Chen-like Attractors [10]



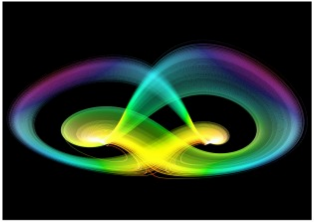
Yet More Lorenz-like to Chen-like Attractors [10]



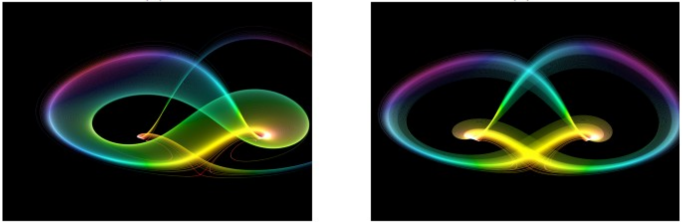
Yet More Lorenz-like to Chen-like Attractors [10]



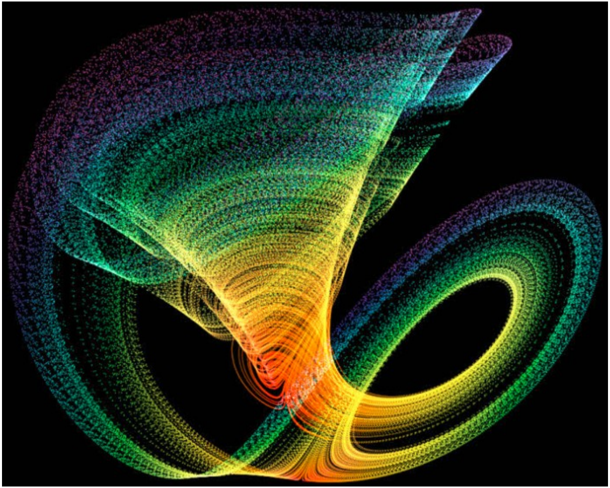
Yet More Lorenz-like to Chen-like Attractors [10]



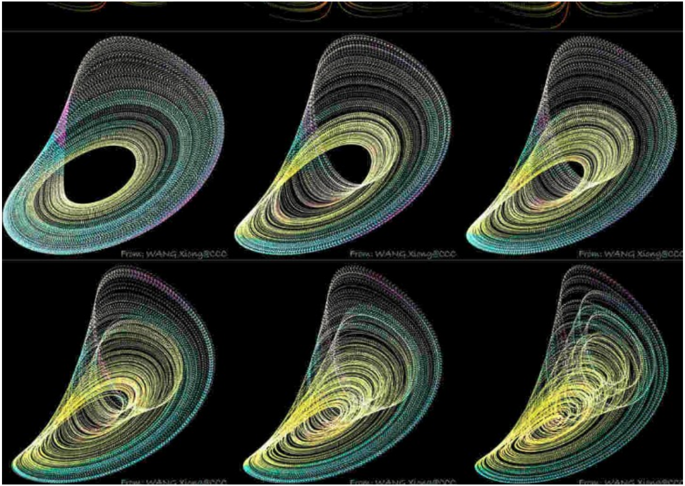
(iii) (iv)



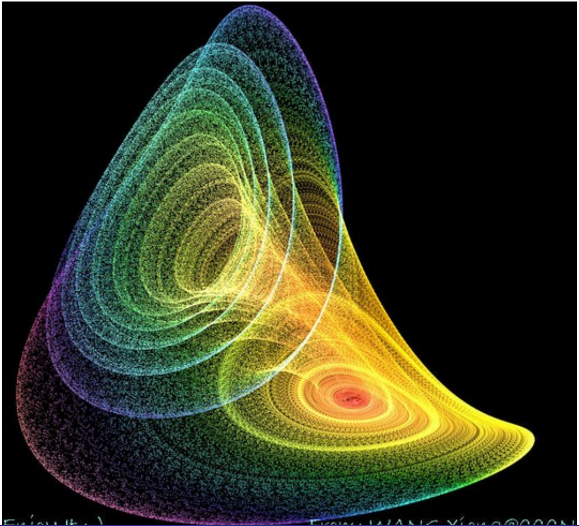
How about asymmetrical attractor?



How about Rössler-like attractor?



Especially for super-twirling attractor?



1 Introduction
2 Knot Theory
3 Nonlinear Dynamics
4 Topology in Chaos
5 Open Questions
6 Summary

Summary

1. Topology is the mathematical study of shapes and spaces.
2. Dynamical Systems Theory deals with the long-term qualitative behavior of dynamical systems.
3. Periodic orbits of a third-order ODE are topological knots. This simple but crucial observation connects these two important areas of mathematical science.
4. The study of the topology of the knots appearing in dynamical systems could lead to better understanding of general dynamical systems.
5. Still many many open problems...

Interesting Links

A walk through mathematics http://www.dimensions-math.org/
CHAOS A Mathematical Adventure http://www.chaos-math.org/en
Beyond Lorenz: Chen's chaotic attractor family
http://www.youtube.com/watch?v=owwI2ICB-G4
Wang's strange attractor blooming process
http://www.youtube.com/watch?v=eNmr-fewDi4
http://www.youtube.com/watch?v=GYtPTW2-D8Y
http://www.youtube.com/watch?v=HOZwIHlqWd4
http://www.youtube.com/watch?v=6N9jrfrBLPA
http://www.youtube.com/watch?v=T7oquYUIBDA
Lorenz knot http://www.youtube.com/watch?v=6MaQP2CMSQQ
http://www.youtube.com/watch?v=v8lp8bTYJU0
http://www.youtube.com/watch?v=UEw4T4RHdos
http://www.youtube.com/watch?v=7h4VI_oEOkM
http://www.youtube.com/watch?v=zbBuxHWu7YE

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Thank You!! Q&A