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COEXISTENCE OF POINT, PERIODIC AND STRANGE ATTRACTORS

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For a dynamical system described by a set of autonomous ordinary differential equations, an attractor can be a point, a periodic cycle, or even a strange attractor. Recently, a new chaotic system with only one stable equilibrium was described, which locally converges to the stable equilibrium but is globally chaotic. This paper further shows that for certain parameters, besides the point attractor and chaotic attractor, this system also has a coexisting stable limit cycle, demonstrating that this new system is truly complicated and interesting.

Keywords: Stable equilibrium; periodic solution; strange attractor.

1. Introduction

Many dynamical systems in the physical world are by nature dissipative. Such dissipation may come from internal friction, thermodynamic loss, energy or material loss, among many causes. Orbits of a dissipative dynamical system will shrink into zero-volume subsets in the state space as time goes to infinity.

For a dynamical system described by a set of autonomous ordinary differential equations (ODEs), $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in R^n$, if $f(\mathbf{x}_e) = 0$ has a real solution, then $\mathbf{x}_e$ is called an *equilibrium* of the dynamical system.

An equilibrium is said to be *hyperbolic* if all eigenvalues of the system's Jacobian matrix evaluated at the equilibrium have nonzero real parts. The signs of these real parts of the eigenvalues determine the stability of the equilibrium. The Hartman–Grobman Theorem [Teschl, 2012] states that the behavior of a dynamical system near a hyperbolic equilibrium is topologically equivalent to (i.e. qualitatively the same as) the behavior of its linearized model in a neighborhood of the equilibrium. Thus, if the equilibrium is stable, it will be a point attractor of the system, which attracts all nearby orbits.

Besides the zero-dimensional point attractors, there are also one-dimensional periodic-cycle attractors, called *limit cycles*, in which an orbit circles around in the state space. Although point attractors and limit cycles are the most common attractors with integer dimension and regular structure, attractors can also be complicated point sets with fractal structure. An attractor is said to be *strange* if it has a noninteger dimension, and examples of such strange attractors are manifest [Sprott, 1993, 1994, 1997; Sprott & Linz, 2000; Lorenz, 1963; Chen & Ueta, 1999; Ueta & Chen, 2000].

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Most systems with strange attractors have at least one unstable equilibrium. However, in addition to the aforementioned Sprott systems [Sprott, 1993, 1994, 1997; Sprott & Linz, 2000], the Lorenz system [Lorenz, 1963] and the Chen system [Chen & Ueta, 1999; Ueta & Chen, 2000] both have two unstable saddle-foci and one unstable node, which can generate a two-wing butterfly-shaped strange attractor, usually referred to as a *chaotic attractor*, for its special properties characterized by sensitive dependence on initial conditions. Of course, it is known that there are also strange but nonchaotic attractors, depending on the definitions used.

An interesting question is whether a simple system (say, one that is three-dimensional and autonomous with only quadratic nonlinearities) can have all three of these attractors concurrently. What follows is just such an example.

2. Coexistence of Point, Periodic and Strange Attractors

A chaotic system with only one equilibrium, a stable node-focus, was introduced in [Wang & Chen, 2012]. This system was found by adding a nonzero constant a to case E in [Sprott, 1994] as follows:

$$\begin{cases} \dot{x} = yz + a \\ \dot{y} = x^2 - y \\ \dot{z} = 1 - 4x, \end{cases} \tag{1}$$

when $a \neq 0$, the stability of the single equilibrium is fundamentally altered. Specifically, when $a > 0$, system (1) possesses only one stable equilibrium:

$$P(x_E, y_E, z_E) = \left(\frac{1}{4}, \frac{1}{16}, -16\,a\right). \tag{2}$$

Interestingly, this stable equilibrium can coexist peacefully with a strange attractor, as reported in [Wang & Chen, 2012]. This means that both a point attractor and a strange attractor dominate the system dynamics in a region of the state space, so it is easy to imagine that there should be an unstable boundary between the two attractors. Will these two basins of attraction have a smooth boundary, or will they be intertwined in a fractal or other type of complicated manner? The following discovery makes this question even more fascinating and harder to answer.

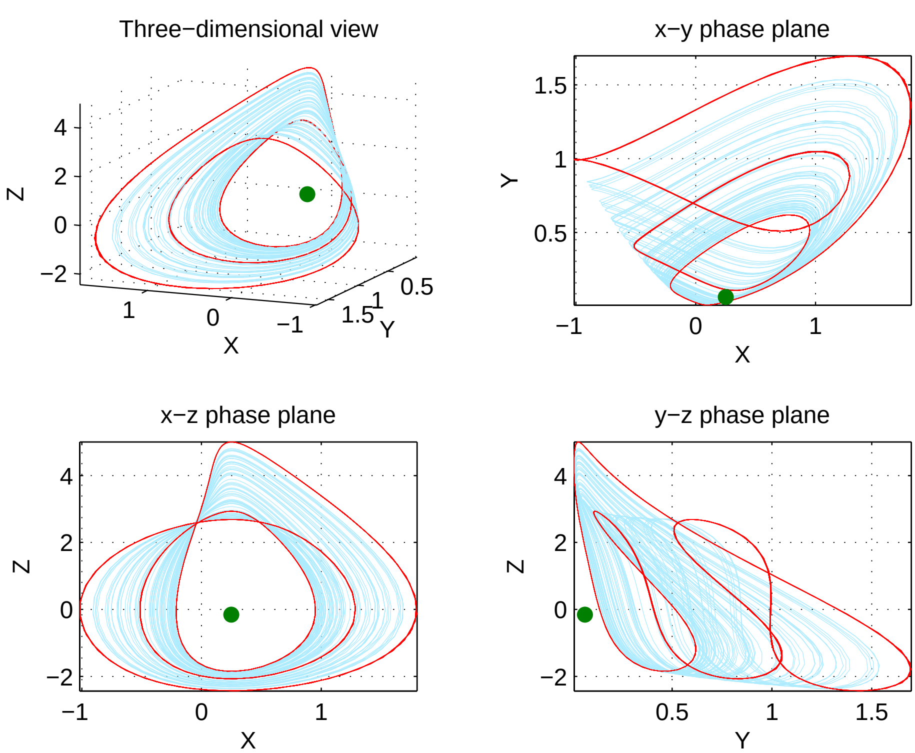

Fig. 1. Coexistence of point, periodic, and strange attractors of system (1) with $a = 0.01$; the point attractor (green) is generated from initial conditions $(0.2, 0, 0)$, the periodic attractor (red) from initial conditions $(1, 1, 1)$, and the strange attractor (blue) from initial conditions $(1, 1, 0)$.

In addition to having a point attractor and a strange attractor, system (1) is now found to have a periodic attractor as well when a is in the vicinity of 0.01, as shown in Fig. 1, giving rise to the coexistence of point, periodic and strange attractors. The point attractor (green) is generated from

Table 1. Lyapunov exponents with different initial values.

Initial Conditions	Lyapunov Exponents	Dimension
(0.2, 0, 0)	-0.033; -0.033; -0.933	0
(1, 1, 1)	0.000; -0.071; -0.929	1
(1, 1, 0)	0.060; 0.000; -1.060	2.057

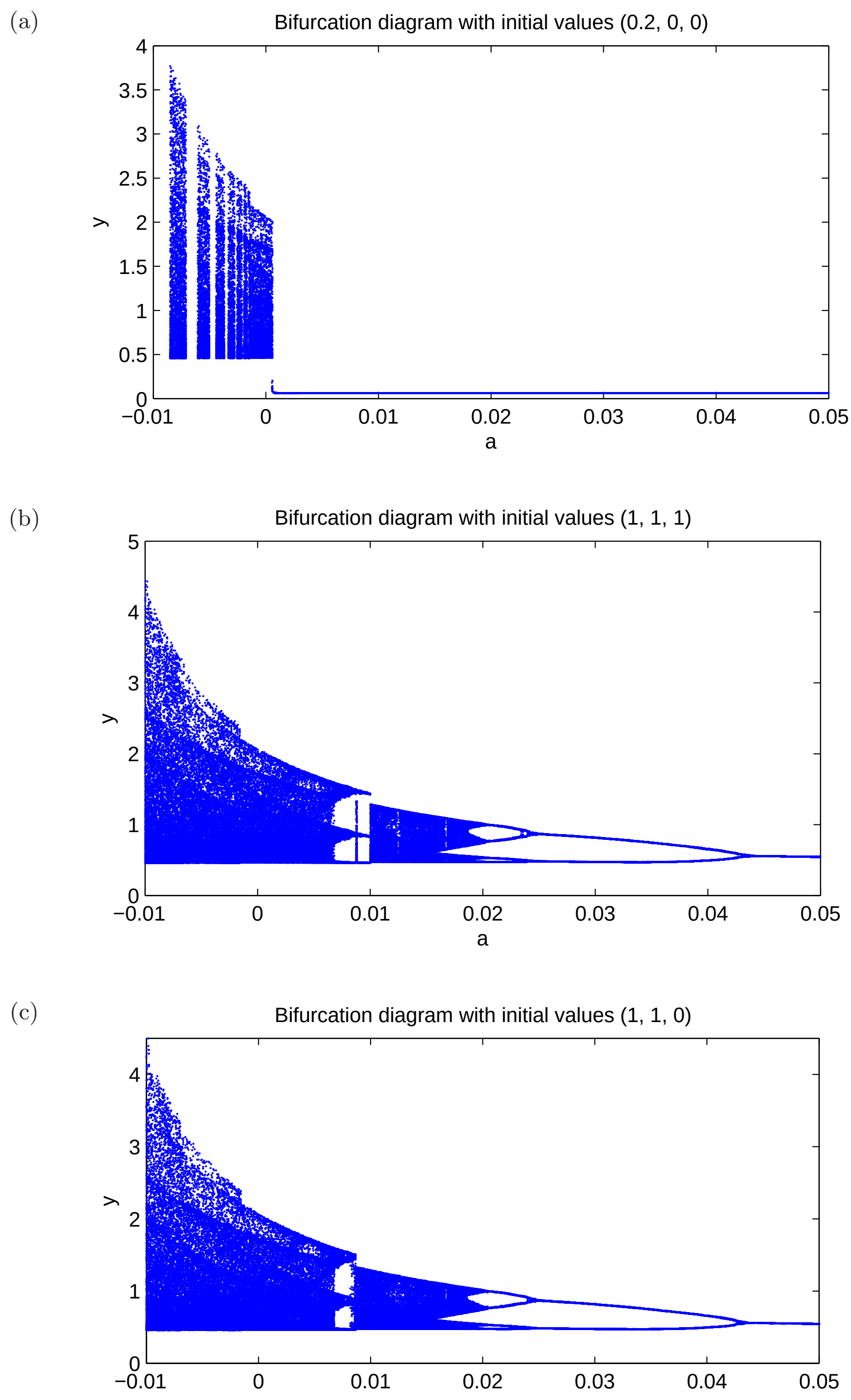

Fig. 2. Bifurcation diagram versus parameter a, with different initial conditions, showing a period-doubling route to chaos.

initial conditions $(x_0, y_0, z_0) = (0.2, 0, 0)$, the periodic attractor (red) from initial conditions $(1, 1, 1)$, and the strange attractor (blue) from initial conditions $(1, 1, 0)$.

The Lyapunov exponents for $a = 0.01$ accurate to three digits are shown in Table 1.

Figure 2 shows the bifurcation diagrams versus parameter a with different initial conditions, demonstrating a period-doubling route to chaos. These diagrams also show that, at $a = 0.01$, the three different initial conditions lead to three different attractors.

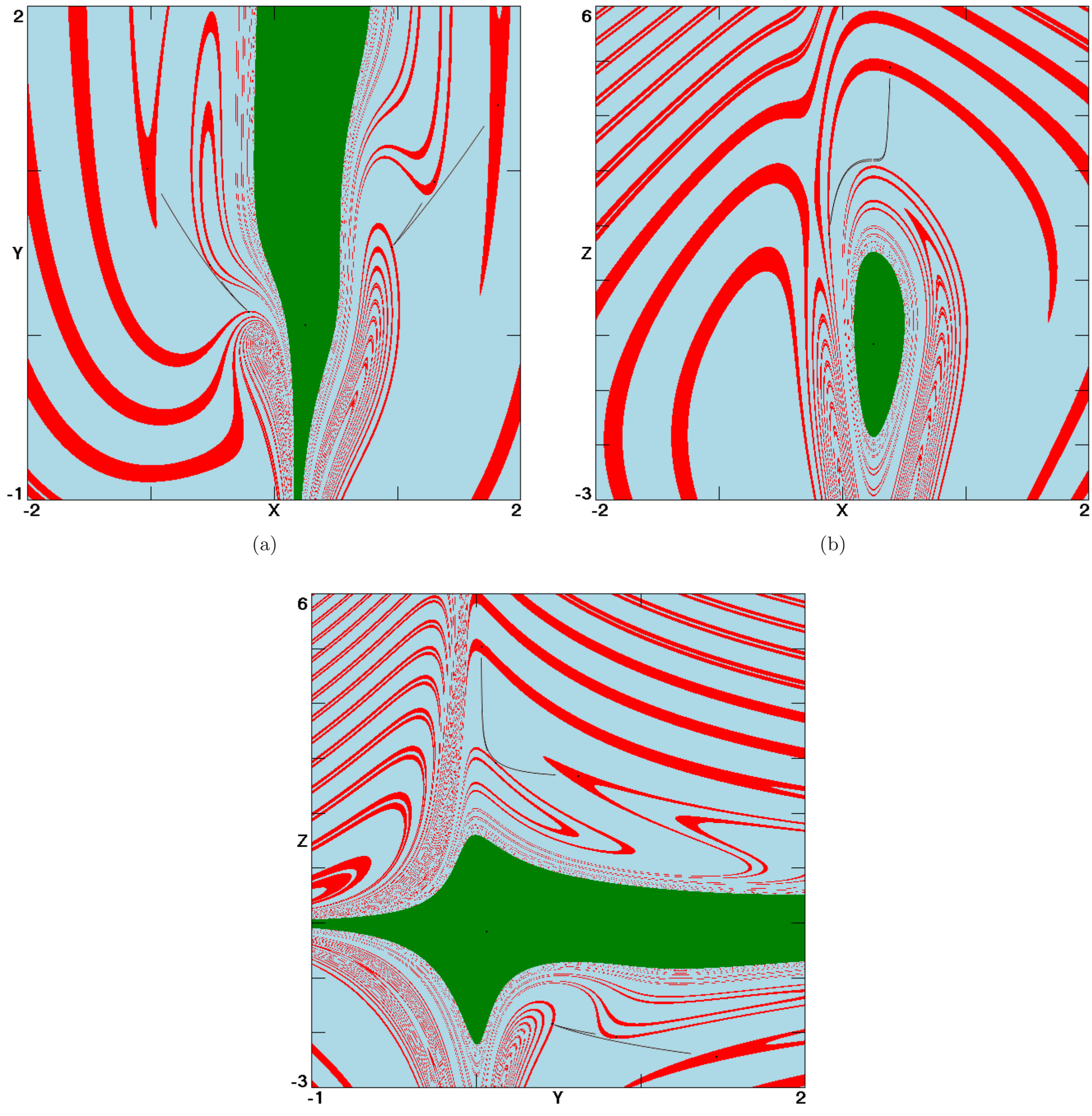

Fig. 3. Basins of attraction of the point, periodic, and strange attractors of system (1), all with $a = 0.01$ on three cross-sections in the plane containing the equilibrium point, marked by green, red, and blue, respectively. The strange attractor resides in the blue basin; the periodic cycle has several points in the red basin, and the equilibrium point is a single point in the green region. The black points are cross-sections of the attractors.

3. Basins of Attraction

The three different types of attractors coexist peacefully in this simple system, with each dominating the dynamics in a different part of the state space. Their basins of attraction represent a mathematically-involved subtle issue, because it is well known that even for multiple point attractors, the basin boundaries can be fractal.

It turns out that the boundaries of the basins of attraction of system (1) do have a fractal structure, three cross-sections of which in the planes containing the equilibrium point are shown in Fig. 3. On these sections, the basins of attraction of the point, periodic, and strange attractors of system (1) with $a = 0.01$ are indicated by green, red, and blue, respectively. The strange attractor resides in the blue basin; the periodic cycle has several points in the red basin, and the equilibrium has a single point in the green region. In this figure, the black points are cross-sections of the respective attractors. As the parameter a is gradually changed, the basins of attraction also gradually change, which makes the estimate of the basin boundaries difficult but interesting.

4. Discussions

An attractor is defined as the smallest attracting point set that cannot be itself decomposed into two or more subsets with distinct basins of attraction. This restriction is necessary since a dynamical system may have different types of multiple attractors, each with its own basin of attraction.

Most systems have only one attractor or one single type of attractor. Others may have two different types of coexisting attractors, most likely strange attractors and periodic cycles. It is interesting and striking to see that the simple system reported here has all three different common types of attractors coexisting side by side. We do not have a definite answer to the question about the mechanism for the birth and death of these different types of attractors, except to note that classical local analytic theory does not apply because the unique equilibrium point of the system is not hyperbolic. One must then resort to the theories of global bifurcation and chaos [Wiggins, 1988], which leaves an important yet challenging theoretical as well as technical problem for future research.

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