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A SIMPLE YET COMPLEX ONE-PARAMETER FAMILY OF GENERALIZED LORENZ-LIKE SYSTEMS

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This paper reports the finding of a simple one-parameter family of three-dimensional quadratic autonomous chaotic systems. By tuning the only parameter, this system can continuously generate a variety of cascading Lorenz-like attractors, which appears to be richer than the unified chaotic system that contains the Lorenz and the Chen systems as its two extremes. Although this new family of chaotic systems has very rich and complex dynamics, it has a very simple algebraic structure with only two quadratic terms (same as the Lorenz and the Chen systems) and all nonzero coefficients in the linear part being −1 except one −0.1 (thus, simpler than the Lorenz and Chen systems). Surprisingly, although this new system belongs to the Lorenz-type of systems in the classification of the generalized Lorenz canonical form, it can generate not only Lorenz-like attractors but also Chen-like attractors. This suggests that there may exist some other unknown yet more essential algebraic characteristics for describing general three-dimensional quadratic autonomous chaotic systems.

Keywords: Chaotic attractor; Lorenz system; Chen system; generalized Lorenz canonical form.

1. Introduction

The scientific story of chaos dates back to earlier 1960s [Lorenz, 1963], when Lorenz studied the atmospheric convection phenomenon he found a chaotic attractor in a three-dimensional (3D) quadratic autonomous system [Sparrow, 1982]. The now-classic Lorenz system is described by

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy, \end{cases} \tag{1}$$

which is chaotic when $\sigma = 10, r = 28, b = \frac{8}{3}$.

Ever since then, many researchers were wondering and pondering whether the discovery of

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the Lorenz system was just a lucky incident or there actually exist other closely related systems around it?

In 1999, from an engineering feedback anticontrol approach, Chen provided a certain answer to this question by finding of a new system [Chen & Ueta, 1999; Ueta & Chen, 2000], lately referred to as the Chen system by others, described by

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = -bz + xy, \end{cases} \tag{2}$$

which is chaotic when $a = 35, b = 3, c = 28$. Whereafter, it has been proved that Chen's attractor exists [Zhou *et al.*, 2004b] and the Chen system displays even more sophisticated dynamical behaviors than the Lorenz system [Zhou *et al.*, 2003].

It is also interesting to recall a unified chaotic system, which was constructed to encompass both the Lorenz system and the Chen system [Lü *et al.*, 2002]. This unified chaotic system is by nature a convex combination of the two systems, and is described by

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x) \\ \dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y \\ \dot{z} = -\dfrac{\alpha + 8}{3}z + xy. \end{cases} \tag{3}$$

When $\alpha = 0$, it is the Lorenz system while with $\alpha = 1$, it is the Chen system, and moreover for any $\alpha \in (0, 1)$ the system remains to be chaotic.

As a result of several years of continued research endeavor along the same line, a family of generalized Lorenz systems was found and characterized [Čelikovský & Chen, 2002a, 2002b, 2005], which is defined through the so-called generalized Lorenz canonical form, as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}, \tag{4}$$

where x, y, z are the state variables of the system. This canonical form contains a family of chaotic systems with the same nonlinear terms, the same symmetry about the z-axis, the same stability of three equilibria, and similar attractors in shape with a two-scroll and butterfly-like structure [Sprott, 1993, 1994, 1997; Sprott & Linz, 2000; Qi *et al.*, 2005; Zhou *et al.*, 2004a, 2006].

According to Čelikovský and Vaneček [1994] (see also [Vaneček & Čelikovský, 1996]), the Lorenz system satisfies $a_{12}a_{21} > 0$, while the Chen system was found to satisfy $a_{12}a_{21} < 0$. In this sense, the Chen system is *dual* to the Lorenz system. In between, it was also found that there is a transition, the Lü system, which satisfies $a_{12}a_{21} = 0$ [Lü & Chen, 2002].

Lately, another form of a unified Lorenz-type system and its canonical form were developed, which contain some generalized Lorenz-type systems and their corresponding conjugate Lorenz-type systems [Yang *et al.*, 2006, 2007].

Moreover, another classification was developed in [Yang & Chen, 2008], where system (3) is classified into two groups, the Lorenz system group if $a_{11}a_{22} > 0$ and the Chen system group if $a_{11}a_{22} < 0$. This classification leads to the finding of a transition, the Yang–Chen chaotic system, which satisfies $a_{11}a_{22} = 0$.

Comparing these two classifications, one can see that each such system is classified according to its algebraic structure. Specifically, it is determined either by $a_{12}a_{21}$ or by $a_{11}a_{22}$.

Given the above discussions, at this point, it is interesting to ask the following questions:

(1) Concerning the conditions on the signs of $a_{12}a_{21}$, defined in [Čelikovský & Vaneček, 1994] (see also [Vaneček & Čelikovský, 1996]), are these signs essential to the system dynamics? For instance, can a system belonging to the Lorenz systems family generate Chen-like attractors?
(2) Concerning the conditions on the signs of $a_{11}a_{22}$, defined in [Yang & Chen, 2008], are these signs essential to the system dynamics? For instance, can a system belonging to the Chen system group generate Lorenz-like attractors?
(3) Concerning the unified chaotic system (3), is there any other simpler chaotic system that can also generate both Lorenz-like attractor and Chen-like attractor?

This paper aims to provide certain answers to the above questions via the finding of a new family of Lorenz-like systems. This new family has only one real parameter in a simple algebraic form but demonstrates very rich and complex dynamics in a way similar to the generalized Lorenz canonical form [Čelikovský & Chen, 2002a, 2002b, 2005]. It belongs to the Lorenz-type of systems defined in [Čelikovský & Vaneček, 1994] and it is also classified into the Chen system group defined in [Yang & Chen, 2008]. However, it has a very simple form and can generate both Lorenz-like attractors and Chen-like attractors by gradually changing the single parameter. This reveals that further study of the system algebraic structure and its effects on the system dynamics remain an important and interesting challenge.

2. The New Family of Chaotic Systems

The new system under investigation is described by

$$\begin{cases} \dot{x} = -x - y \\ \dot{y} = -x + ry - xz \\ \dot{z} = -0.1z + xy, \end{cases} \tag{5}$$

where r is a real parameter.

This is a one-parameter family of chaotic systems in the sense that as the real parameter r is gradually varied, a sequence of chaotic attractors can be continuously generated from the system, with very rich and complicated dynamics, as further demonstrated below.

Before proceeding to chaotic dynamics, some basic dynamical properties of this new system are analyzed following a standard routine, which is necessary for understanding the nature of the new system.

2.1. *Symmetry and dissipativity*

First, it is apparent that system (5) has a natural symmetry about the z-axis under the following transformation:

$$S(x, y, z) \to (-x, -y, z).$$

Second, one may construct the following Lyapunov function:

$$V(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2), \tag{6}$$

which gives

$$\begin{aligned} \dot{V}(x, y, z) &= x\dot{x} + y\dot{y} + z\dot{z} \\ &= x(-x - y) + y(-x + ry - xz) \\ &\quad + z(-0.1z + xy) \\ &= -(x + y)^2 + (r + 1)y^2 - 0.1z^2. \end{aligned} \tag{7}$$

This implies that system (5) is globally uniformly asymptotically stable about its zero equilibrium if $r < -1$. Consequently, system (5) is not chaotic in the parameter region $r < -1$.

Third, since

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = r - 1.1,$$

the system is dissipative under the condition of $r < 1.1$. More precisely, since $V(t) = V(t_0)e^{-(1.1-r)t}$, any initial volume V_0 containing the system trajectories shrinks to zero as $t \to +\infty$ at an exponential rate of $1.1 - r$.

2.2. *Equilibria and their stability*

It is obvious that the origin is a trivial equilibrium of system (5). Other nonzero equilibria can be found by solving the following equations simultaneously:

$$-x - y = 0; \quad -x + ry - xz = 0;$$
$$-0.1z + xy = 0.$$

When $r > -1$, system (5) has three equilibria:

$$O(0, 0, 0), \quad E_1\left(\sqrt{\frac{1+r}{10}}, -\sqrt{\frac{1+r}{10}}, -1 - r\right),$$
$$E_2\left(-\sqrt{\frac{1+r}{10}}, \sqrt{\frac{1+r}{10}}, -1 - r\right).$$

By linearizing system (5) at $O(0, 0, 0)$, one obtains the Jacobian

$$J|_O = \begin{bmatrix} -1 & -1 & 0 \\ -1 - z & r & -x \\ y & x & -0.1 \end{bmatrix}, \tag{8}$$

whose characteristic equation is

$$\begin{aligned} \det(\lambda I - J|_O) &= \lambda^3 + (1.1 - r)\lambda^2 - (1.1r + 0.9)\lambda \\ &\quad - 0.1(r + 1) \\ &= 0, \end{aligned}$$

which yields

$$\lambda_1 = -0.1 < 0,$$
$$\lambda_2 = -0.5 + 0.5r + \frac{\sqrt{5 + 2r + r^2}}{2} > 0,$$
$$\lambda_3 = -0.5 + 0.5r - \frac{\sqrt{5 + 2r + r^2}}{2} < 0.$$

Table 1. Equilibria and eigenvalues of several typical systems.

Systems	Equations	Equilibria	Eigenvalues
Unified system	$\dot{x} = 10(y - x)$	$(0, 0, 0)$	$-22.8277, -2.6667, 11.8277$
$\alpha = 0$	$\dot{y} = 28x - y - xz$		
	$\dot{z} = -\frac{8}{3}z + xy$	$(\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$	$-13.8546, \mathbf{0.0940} \pm 0.1945i$
Unified system	$\dot{x} = \frac{45}{2}(y - x)$	$(0, 0, 0)$	$-28.1696, -2.8333, 19.1696$
$\alpha = 0.5$	$\dot{y} = \frac{21}{2}x + \frac{27}{2}y - xz$		
	$\dot{z} = -\frac{17}{6}z + xy$	$(\pm\sqrt{68}, \pm\sqrt{68}, 24)$	$-16.9593, \mathbf{2.5630} \pm 13.1857i$
Unified system	$\dot{x} = 30(y - x)$	$(0, 0, 0)$	$-30.000, -2.9333, 22.200$
$\alpha = 0.8$	$\dot{y} = \frac{111}{5}y - xz$		
	$\dot{z} = -\frac{44}{15}z + xy$	$\left(\pm\frac{2\sqrt{407}}{5}, \pm\frac{2\sqrt{407}}{5}, 22.2\right)$	$-17.9535, \mathbf{3.6101} \pm 14.3037i$
Unified system	$\dot{x} = 35(y - x)$	$(0, 0, 0)$	$-30.8357, -3, 23.8359$
$\alpha = 1$	$\dot{y} = -7x + 28y - xz$		
	$\dot{z} = -3z + xy$	$(\pm 3\sqrt{7}, \pm 3\sqrt{7}, 21)$	$-18.4288, \mathbf{4.2140} \pm 14.8846i$
New system	$\dot{x} = -x - y$	$(0, 0, 0)$	$-0.1, 0.6544, -1.6044$
$r = 0.05$	$\dot{y} = -x + 0.05y - xz$		
	$\dot{z} = -0.1z + xy$	$\left(\pm\frac{\sqrt{42}}{20}, \mp\frac{\sqrt{42}}{20}, -1.05\right)$	$-1.05, \mathbf{0} \pm 0.4472i$
New system	$\dot{x} = -x - y$	$(0, 0, 0)$	$-0.1, 0.6913, -1.5913$
$r = 0.1$	$\dot{y} = -x + 0.1y - xz$		
	$\dot{z} = -0.1z + xy$	$\left(\pm\frac{\sqrt{11}}{10}, \mp\frac{\sqrt{11}}{10}, -1.1\right)$	$-1.0162, \mathbf{0.0081} \pm 0.4652i$
New system	$\dot{x} = -x - y$	$(0, 0, 0)$	$-0.1, 1, -1.5$
$r = 0.5$	$\dot{y} = -x + 0.5y - xz$		
	$\dot{z} = -0.1z + xy$	$\left(\pm\frac{\sqrt{15}}{10}, \mp\frac{\sqrt{15}}{10}, -1.5\right)$	$-0.8101, \mathbf{0.1051} \pm 0.5994i$
New system	$\dot{x} = -x - y$	$(0, 0, 0)$	$-0.1, 1.1624, -1.4624$
$r = 0.7$	$\dot{y} = -x + 0.7y - xz$		
	$\dot{z} = -0.1z + xy$	$\left(\pm\frac{\sqrt{17}}{10}, \mp\frac{\sqrt{17}}{10}, -1.7\right)$	$-0.7446, \mathbf{0.1723} \pm 0.6534i$
New system	$\dot{x} = -x - y$	$(0, 0, 0)$	$-0.1, 1.2038, -1.4538$
$r = 0.75$	$\dot{y} = -x + 0.75y - xz$		
	$\dot{z} = -0.1z + xy$	$\left(\pm\frac{\sqrt{70}}{20}, \mp\frac{\sqrt{70}}{20}, -1.75\right)$	$-0.7312, \mathbf{0.1906} \pm 0.6651i$

Obviously, the equilibrium $O(0,0,0)$ is a saddle point, at which the stable manifold $W^s(O)$ is two-dimensional and the unstable manifold $W^u(O)$ is one-dimensional.

Similarly, linearizing the system with respect to the other equilibria, E_1 and E_2, yields the following characteristic equation:

$$\det(\lambda I - J|_{E_{1,2}}) = \lambda^3 + (1.1 - r)\lambda^2 + 0.2\lambda + 0.2(r+1) = 0.$$

According to the Routh–Hurwitz criterion, the following constraints are imposed:

$$\Delta_1 = (1.1 - r) > 0, \tag{9}$$

$$\Delta_2 = \begin{vmatrix} 1.1 - r & 0.2(r+1) \\ 1 & 0.2 \end{vmatrix} = 0.1 - 2r > 0, \tag{10}$$

$$\Delta_3 = 0.2(r+1)\Delta_2 > 0. \tag{11}$$

This characteristic polynomial has three roots, all with negative real parts, under the condition of $-1 < r < 0.05$. Therefore, the two equilibria E_1 and E_2 are both stable nodes, or node-foci, if $-1 < r < 0.05$.

However, if the condition $0.05 < r < 1.1$ holds, then the characteristic equation has one negative real root and one pair of complex conjugate roots with a positive real part. Therefore, the two equilibria $E_{1,2}$ are both saddle-foci, at each of which the stable manifold $W^s(E_{1,2})$ is one-dimensional and the unstable manifold $W^u(E_{1,2})$ is two-dimensional.

Summarizing the above analysis and discussions establishes the following result:

Theorem 1. *With parameter $-1 < r < 1.1$, system (5) has three equilibria:*

$$O(0,0,0), \quad E_1\left(\sqrt{\frac{1+r}{10}}, -\sqrt{\frac{1+r}{10}}, -1-r\right),$$

$$E_2\left(-\sqrt{\frac{1+r}{10}}, \sqrt{\frac{1+r}{10}}, -1-r\right).$$

Furthermore,

(i) *$O(0,0,0)$ is a saddle point, at which the stable manifold $W^s(O)$ is two-dimensional and the unstable manifold $W^u(O)$ is one-dimensional;*
(ii) *if $0.05 < r < 1.1$, then the two equilibria E_1 and E_2 are both saddle-foci, at each of which the stable manifold $W^s(E_{1,2})$ is one-dimensional and the unstable manifold $W^u(E_{1,2})$ is two-dimensional;*
(iii) *if $-1 < r < 0.05$, then the equilibria E_1 and E_2 are both stable nodes, or node-foci, at each of which the stable manifold $W^s(E_{1,2})$ is three-dimensional.*

The Jacobian eigenvalues evaluated at the three equilibria of the unified chaotic system (3) and of the new system (5) are listed in Table 1 for comparison.

Given the above analysis, system (5) will be discussed only within the parameter region of $0.05 \le r < 1.1$ below.

2.3. *Remarks on the classification of the new system*

On one hand, it is noted that in system (5), one has $a_{12}a_{21} = 1 > 0$, so it belongs to the Lorenz system family defined in [Čelikovský & Vaneček, 1994].

On the other hand, by making the following transformation:

$$T : (x, y, z) \to (x, -y, -z),$$

system (5) becomes

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = x + ry - xz \\ \dot{z} = -0.1z + xy, \end{cases} \tag{12}$$

which satisfies $a_{11}a_{22} < 0$ and $a_{11} < 0$, thus belonging to the Chen system group according to the classification in [Yang & Chen, 2008] (see Tables 3 and 4 therein).

It is therefore very interesting to see that the new system (5) belongs to the same class (either Lorenz-type of systems defined in [Čelikovský & Vaneček, 1994] or Chen system group defined in [Yang & Chen, 2008]), yet can generate both Lorenz-like attractors and Chen-like attractors by gradually changing the single parameter r from 0.05 to 0.74, as further detailed next.

3. Chaotic Behaviors and Other Complex Dynamics

A complete transition from Lorenz-like to Chen-like attractors in system (5) is shown in Fig. 1. The bifurcation diagram with respect to parameter r is shown in Fig. 2. From these figures, one can observe that system (5) evolves to a chaotic state

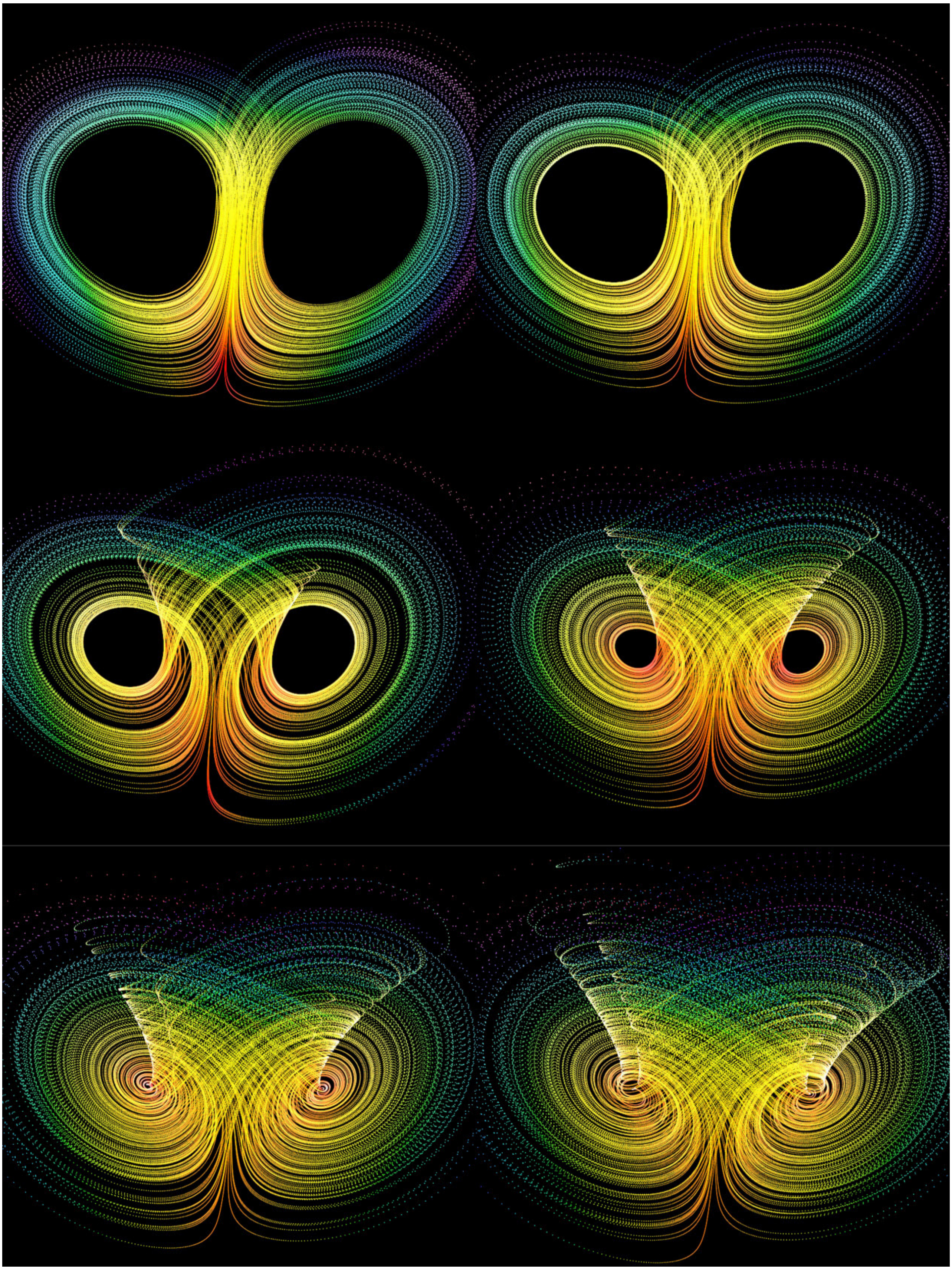


Fig. 1. Colored figures showing the projected orbit transition from Lorenz-like to Chen-like attractors in system (5).

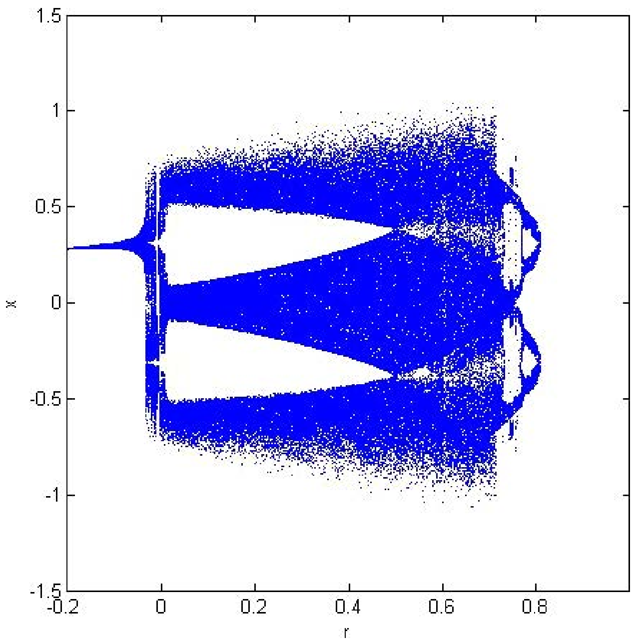


Fig. 2. Bifurcation diagram of system (5) with $r \in [-0.2, 0.8]$.

and eventually to a limit cycle, as the parameter r gradually increases.

To verify the chaotic behaviors of system (5), its three Lyapunov exponents $L_1 > L_2 > L_3$ and

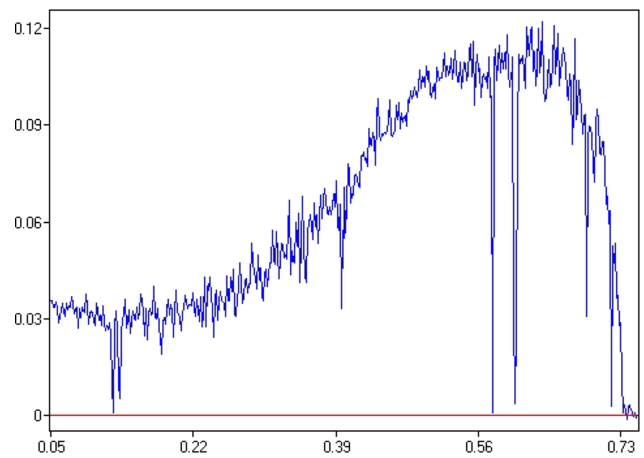


Fig. 3. The largest Lyapunov exponent of system (5) with $r \in [0.05, 0.75]$.

the Lyapunov dimension are calculated, where the latter is defined by

$$D_L = j + \frac{1}{|L_{j+1}|} \sum_{i=1}^{j} L_i,$$

in which j is the largest integer satisfying $\sum_{i=1}^{j} L_i \geq 0$ and $\sum_{i=1}^{j+1} L_i < 0$.

Note that system (5) is chaotic if $L_1 > 0, L_2 = 0, L_3 < 0$ with $|L_1| < |L_3|$. Figure 3 shows the

Table 2. Numerical results for some values of the parameter r with initial values $(1, 1, 1)$.

Parameters	Eigenvalues	Lyapunov Exponents	Fractal Dimensions
$r = 0.05$	$\lambda_1 = -1.05$	$L_1 = 0.03467$	
	$\lambda_{2,3} = \pm 0.4472i$	$L_2 = 0$	$D_L = 2.0317$
		$L_3 = -1.0843$	
$r = 0.1$	$\lambda_1 = -1.0162$	$L_1 = 0.03058$	
	$\lambda_{2,3} = 0.0081 \pm 0.4652i$	$L_2 = 0$	$D_L = 2.029$
		$L_3 = -1.0301$	
$r = 0.5$	$\lambda_1 = -0.8101$	$L_1 = 0.10531$	
	$\lambda_{2,3} = 0.1051 \pm 0.5994i$	$L_2 = 0$	$D_L = 2.1497$
		$L_3 = -0.70561$	
$r = 0.7$	$\lambda_1 = -0.7446$	$L_1 = 0.08953$	
	$\lambda_{2,3} = -0.1723 \pm 0.6534i$	$L_2 = 0$	$D_L = 2.1832$
		$L_3 = -0.48973$	
$r = 0.75$	$\lambda_1 = -0.7312$	$L_1 = 0$	
	$\lambda_{2,3} = 0.1906 \pm 0.6651i$	$L_2 = -0.04626$	$D_L = 1.0063$
		$L_3 = -3.0403$	

dependence of the largest Lyapunov exponent on the parameter r. In particular, for several values of r, the Lyapunov exponents and dimensions of system (5) are summarized in Table 2. From Fig. 3, it is clear that the largest Lyapunov exponent increases as the parameter r increases from 0.05 to 0.7.

***Case 1*: $r = 0.05$**

When $r < 0.05$, the corresponding system has one saddle and two stable node-foci (for reference, see [Yang & Chen, 2008]). The case of $r < 0.05$ is subtle and somewhat complicated, which will be further studied elsewhere in the near future.

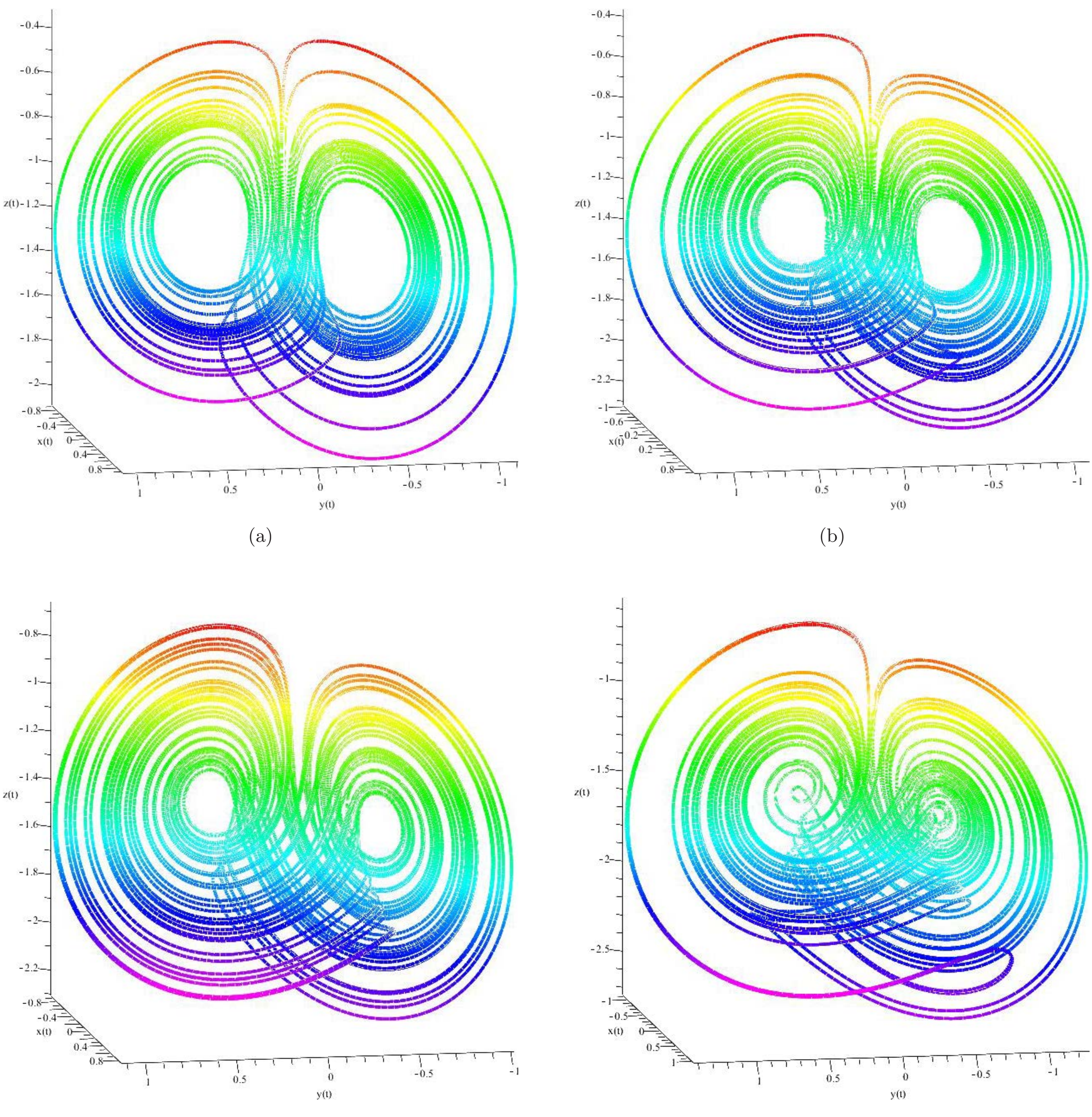


Fig. 4. Attractors of system (5) with parameter values: (a) $r = 0.2$, (b) $r = 0.3$, (c) $r = 0.4$, (d) $r = 0.50$, (e) $r = 0.6$, (f) $r = 0.7$, (g) $r = 0.72$, (h) $r = 0.75$.

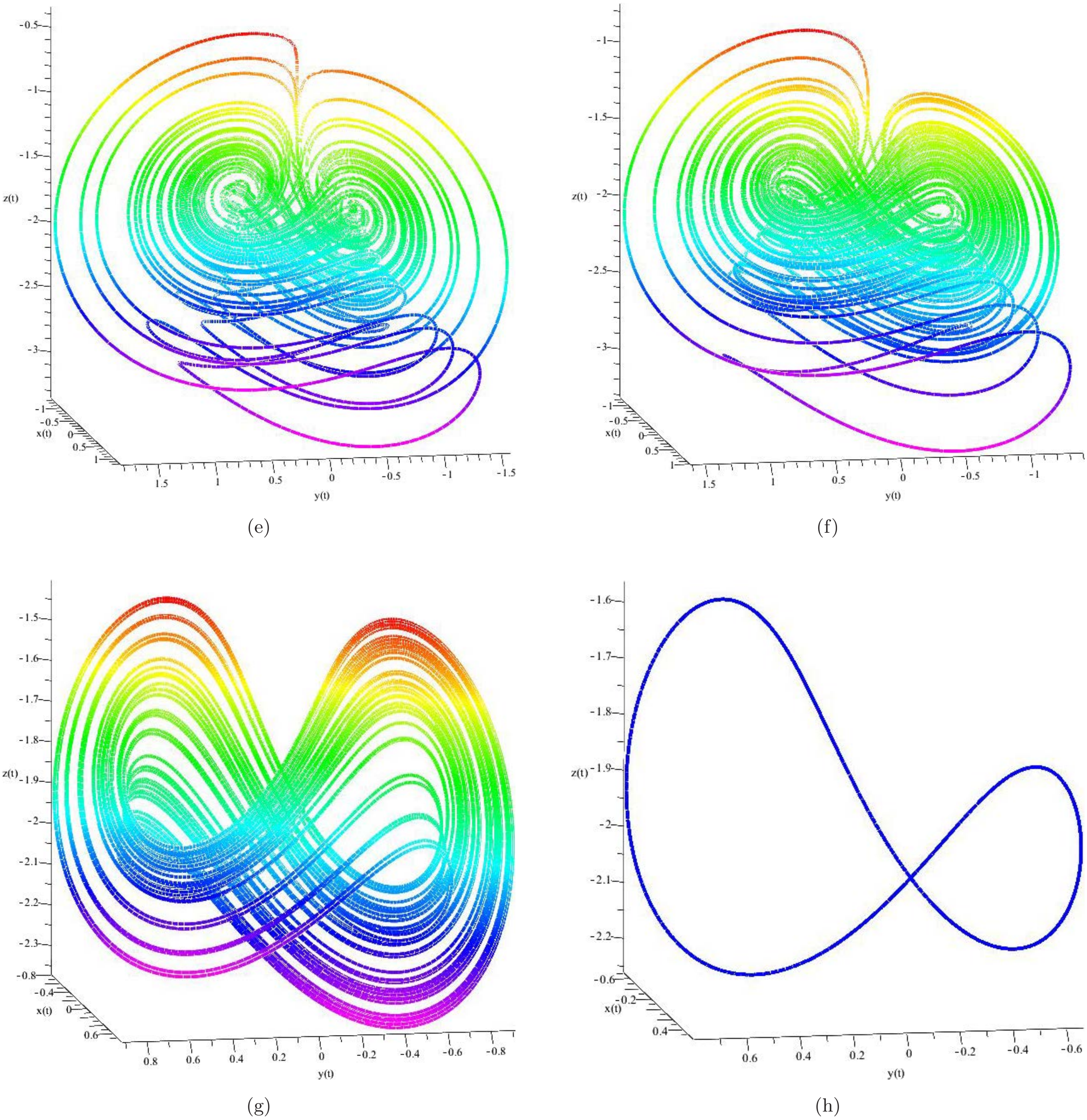


Fig. 4. (*Continued*)

Here, the discussion starts from $r = 0.05$. When $r = 0.05$, this new system has one saddle and two centers. The two centers have the same set of eigenvalues: one negative real eigenvalue and two conjugate complex eigenvalues with zero real parts. Nevertheless, the Lyapunov exponents are $L_1 = 0.03467$, $L_2 = 0$, and $L_3 = -1.0843$, and the Lyapunov dimension is $D_L = 2.0317$, for initial values $(1, 1, 1)$. This convincingly implies that the system is chaotic.

Case 2: $r \in (0.05, 0.7]$

In this case, the new system has one saddle and two saddle-foci. The two saddle-foci have one negative real eigenvalue and two conjugate complex eigenvalues with a positive real part. Moreover, the largest

Lyapunov exponent is larger than zero, which is increased as the parameter r is increased from 0.05 to 0.7. At the same time, the attractor of this system is changed from Lorenz-like to Chen-like as the parameter r is increased from 0.05 to around 0.7, as seen in Figs. 4(a)–4(f).

Case 3: $r \in (0.7, 0.74)$

When r lies in this interval, system (5) is still chaotic, for Fig. 3 shows that the largest Lyapunov exponent is positive. However, the attractor is neither Lorenz-like nor Chen-like, as shown in Fig. 4(g), when $r = 0.72$. Moreover, the chaotic property becomes less significant as compared with Case 2. Indeed, the largest Lyapunov exponent decreases when the parameter is increased from $r = 0.7$ to $r = 0.74$, as shown in Fig. 3.

Case 4: $r \in [0.74, 1]$

In this case, although the new system has one saddle and two saddle-foci, associated with one negative real eigenvalue and two conjugate complex eigenvalues with a positive real part, the system is not chaotic. Instead, the system has a limit cycle, though different from Cases 2 and 3. Figure 4(h) displays the limit cycle when $r = 0.75$.

4. Comparison Between the New Family and the Unified System

It is interesting to compare the new family of chaotic systems with the now-familiar generalized Lorenz systems family, particularly the corresponding Lorenz-type and Chen-type systems, as shown in Figs. 6 and 7.

Recall in particular the unified chaotic system (3), which is chaotic for all $\alpha \in [0, 1]$, with a positive largest Lyapunov exponent as shown in Fig. 5.

There are several common features between the new family of chaotic systems (5) and the unified chaotic system (3):

(1) They are both one-parameter family of chaotic systems, representing infinitely many non-equivalent chaotic systems.
(2) They both demonstrate a continuously changing process generating from Lorenz-like to Chen-like attractors.
(3) They both have a similar simple algebraic structure with two same nonlinear terms (i.e. $-xz$ in the second equation and xy in the third equation), just like the Lorenz and the Chen systems.
(4) They both have the same z-axis rotational symmetry, therefore all their attractors are visually similar in shape with a two-scroll and butterfly-like structure.
(5) They both have three equilibria: one saddle and two saddle-foci. Moreover, the two saddle-foci have the same eigenvalues: one negative real number and two conjugate complex numbers with a positive real part. Specifically, for the unified chaotic system, the positive real part of the conjugate complex eigenvalues increases

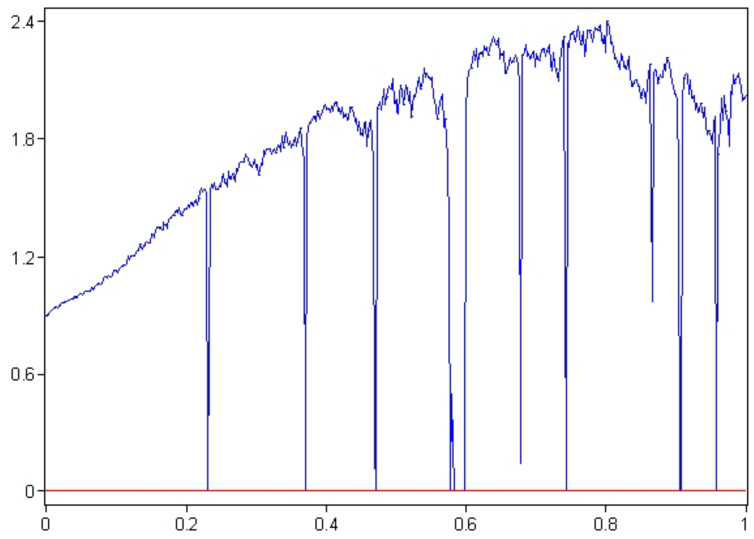


Fig. 5. The largest Lyapunov exponent of the unified chaotic system with $\alpha \in [0, 1]$.

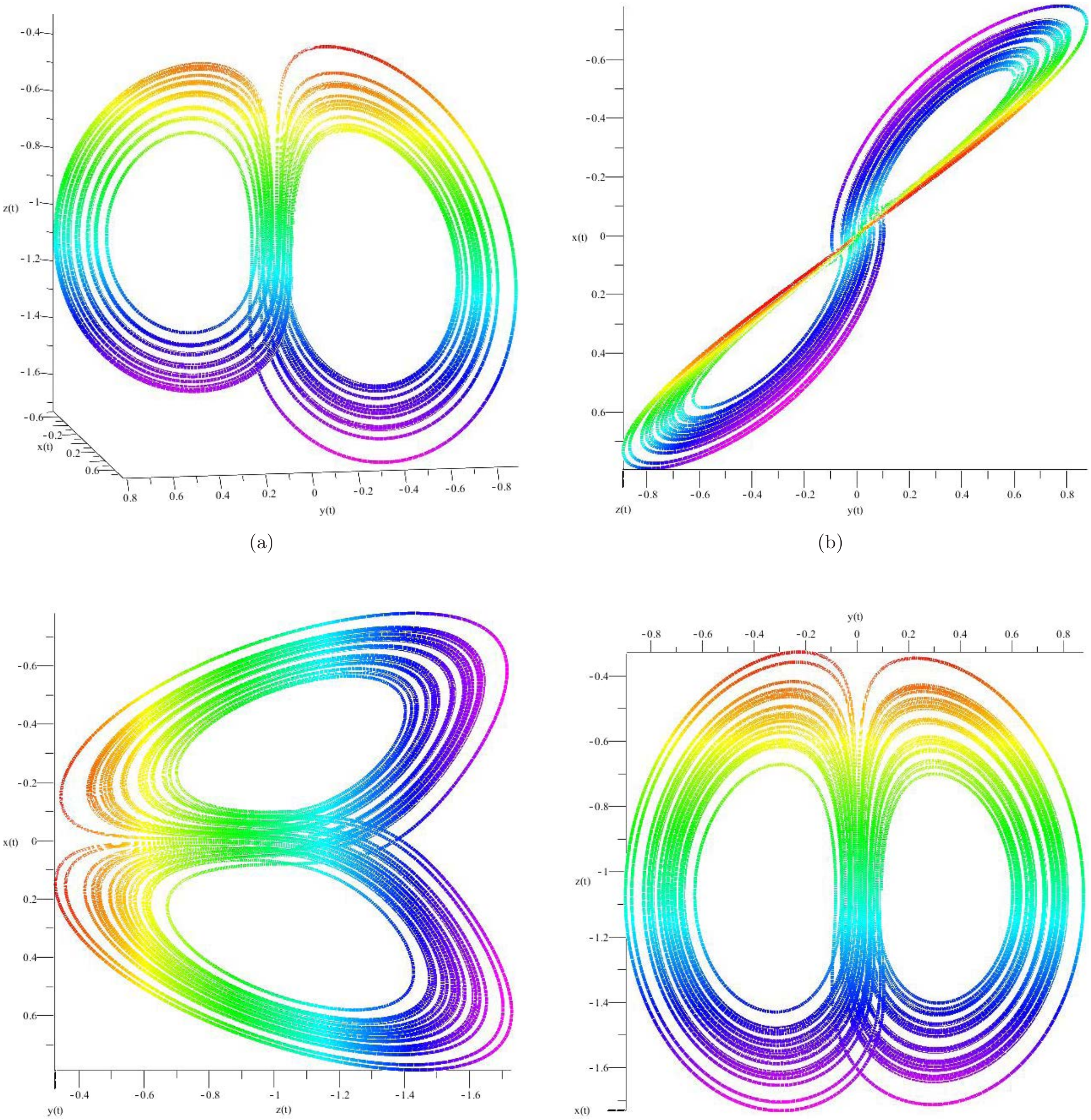


Fig. 6. Attractor of system (5) with $r = 0.06$: (a) the 3D phase portrait, (b) projected orbit on x–y plane, (c) projected orbit on x–z plane, (d) projected orbit on y–z plane. The Lorenz attractor corresponding to $\alpha = 0$ in the system (3), (e) the 3D phase portrait, (f) projected orbit on x–y plane, (g) projected orbit on x–z plane, (h) projected orbit on y–z plane.

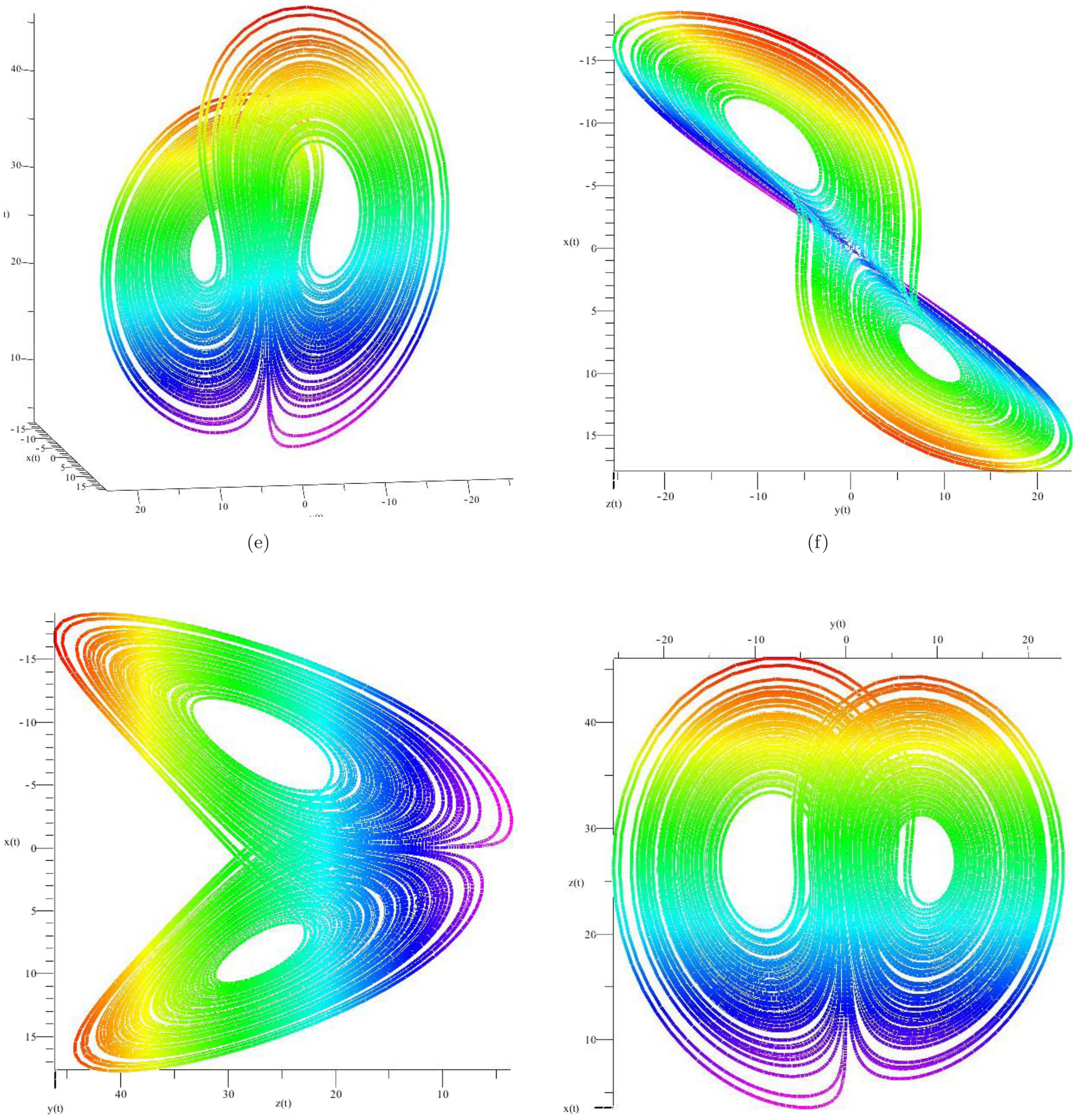


Fig. 6. (*Continued*)

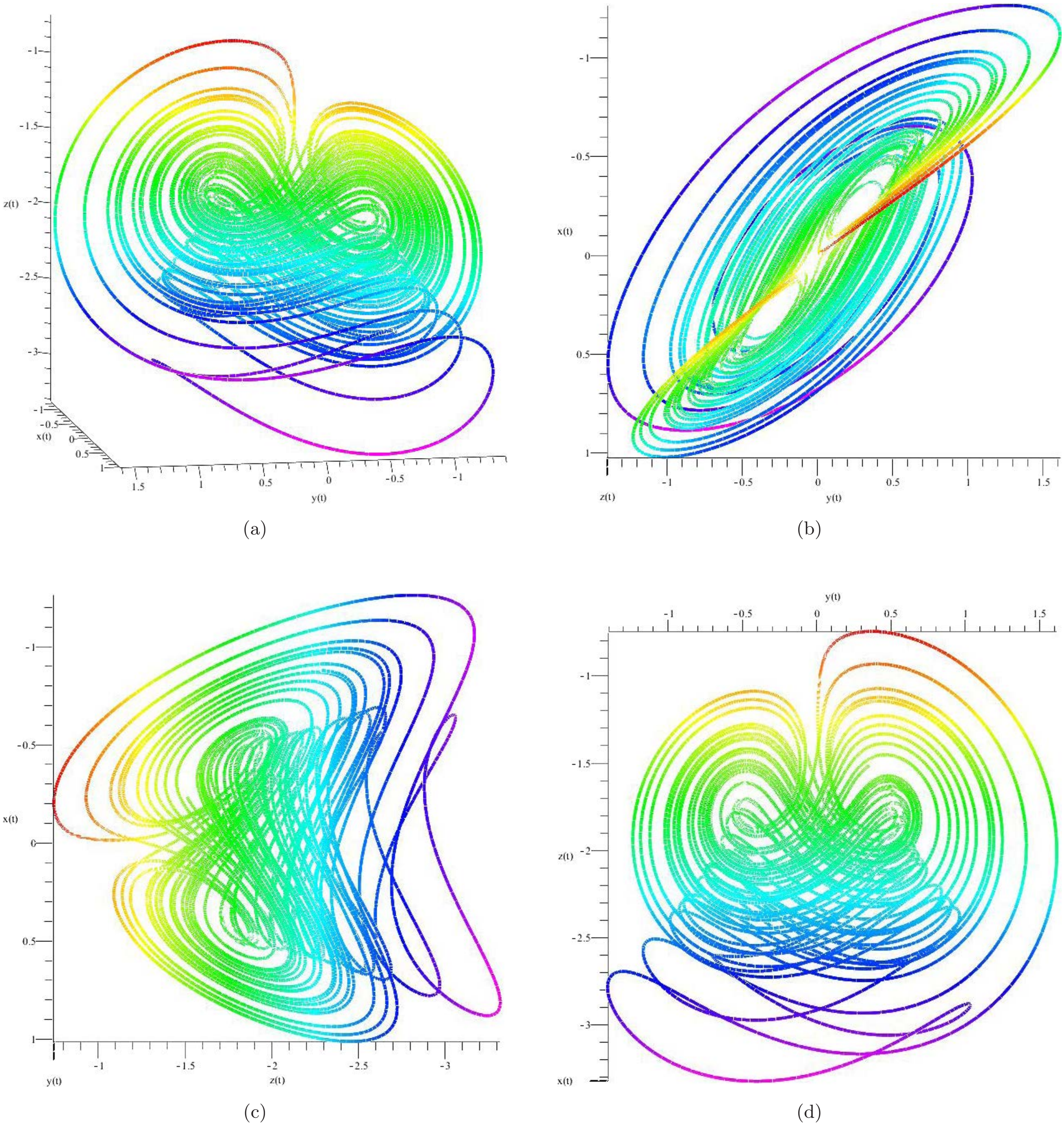
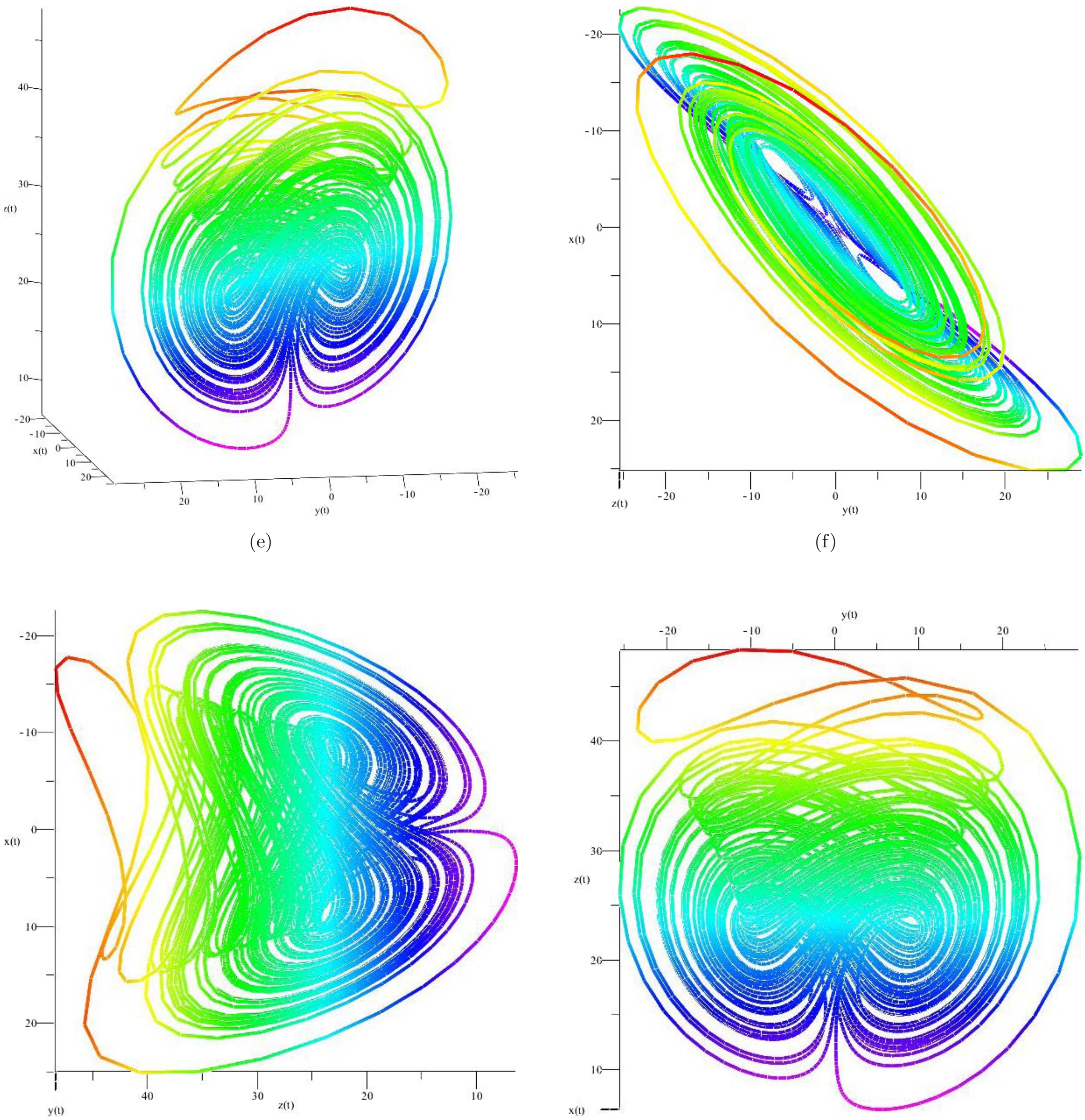


Fig. 7. Attractor of system (5) with $r = 0.7$: (a) the 3D phase portrait, (b) projected orbit on x–y plane, (c) projected orbit on x–z plane, (d) projected orbit on y–z plane. The Chen attractor corresponding to $\alpha = 1$ in the system (3), (e) the 3D phase portrait, (f) projected orbit on x–y plane, (g) projected orbit on x–z plane, (h) projected orbit on y–z plane.



(e) (f) (g) (h)

Fig. 7. (*Continued*)

from the Lorenz system (0.0940) to the Chen system (4.2140), while for the new system, it increases as r is increased from 0.05 to 0.7.

The main difference between these two one-parameter families of chaotic systems are quite subtle. For system (5), the real part of its conjugate complex eigenvalues is precisely zero when $r = 0.05$. This is a critical point when the stability of the equilibrium pairs is changed from stable to unstable. Therefore, one may define the value of $r = 0.05$ as a boundary, or more precisely as a starting point of the Lorenz systems family. Since the parameter r can take values on both sides of $r = 0.05$ to generate chaos, the new family system (5) is richer and more interesting than the unified system (3).

5. Concluding Remarks and Future Research

5.1. *Concluding remarks*

Regarding the few questions raised in the Introduction section, we now come out with some answers:

(1) In this paper, we have reported the finding of a structurally simple yet dynamically complex one-parameter family of 3D autonomous systems with only two quadratic terms like the Lorenz system, with some key constant coefficients set as 1 or -1, leaving only one tunable real parameter in the linear part of the system.
(2) By tuning the singe parameter, this new family of systems can generate a chain of chaotic attractors in a very similar gradual changing process like the one from Lorenz attractor to Chen attractor generated by the generalized Lorenz systems family and the one by the unified chaotic system.
(3) Unlike the typical Chen system, however, the new system that generates the Chen-like attractor does not satisfy the condition $a_{12}a_{21} < 0$ in the linear part of the generalized Lorenz canonical form defined in [Čelikovský & Vaneček, 1994].
(4) Unlike the classical Lorenz system, the new system that can generate a similar Lorenz-like attractor does not satisfy the condition $a_{11}a_{22} > 0$ in the linear part of the generalized Lorenz canonical form defined in [Yang & Chen, 2008].
(5) This system is richer and more interesting than the unified chaotic system, in the sense that the new single parameter has a wider range to generate similar form of chaos.
(6) Similar to the generalized Lorenz systems family, the new system also has three equilibria, with one saddle and two saddle-foci, where the two saddle-foci have the same eigenvalues (one is negative real and two are complex conjugate with a positive real part). However, in the new system the real part of the eigenvalues starts from zero, which can be used to define a starting point for the Lorenz-like systems family. In this sense, the new system has literally extended the unified chaotic system, even the generalized Lorenz systems family, to some extent.

5.2. *Future research*

Some related research issues may be further pursued in the near future:

(1) There may exist other Lorenz-type systems with the same nonlinear terms as the classical Lorenz system, which can generate a variety of chaotic attractors.
(2) There may even exist other types of nonlinear terms that satisfy the expected z-axis rotational symmetry, for which the nonlinear term in the second equation must be like xz or yz, and that in the third equation must be like xy, or z^2, or x^2, or y^2. So, if one wants to study 3D autonomous systems with two quadratic terms that can maintain the z-axis rotational symmetry, then there are totally $2 \times 4 = 8$ possible combinations to consider. It is therefore interesting to ask if there would be other chaotic systems with two nonlinear terms different from those studied so far around the Lorenz system.
(3) In the new family of chaotic systems studied in this paper, the real part of the complex conjugate eigenvalues at system equilibria starts from zero for the case with $r = 0.05$, which can be defined as the starting point of the Lorenz-like systems family. This particularly interesting critical system with $r = 0.05$ alone deserves further investigation.
(4) This paper has further revealed that some systems with different structures can literally generate similar-shaped chaotic attractors. Therefore, for 3D autonomous systems with two quadratic terms, the relation between the system algebraic structure and system chaotic

dynamics is an important and interesting issue to be further revealed, understood and analyzed.

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References

Čelikovský, S. & Chen, G. [2002a] "On a generalized Lorenz canonical form of chaotic systems," *Int. J. Bifurcation and Chaos* **12**, 1789–1812.

Čelikovský, S. & Chen, G. [2002b] "Hyperbolic-type generalized Lorenz system and its canonical form," *Proc. 15th Triennial World Congress of IFAC* Barcelona, Spain, July 2002, in CD ROM.

Čelikovský, S. & Chen, G. [2005] "On the generalized Lorenz canonical form," *Chaos Solit. Fract.* **26**, 1271–1276.

Čelikovský, S. & Vaneček, A. [1994] "Bilinear systems and chaos," *Kybernetika* **30**, 403–424.

Chen, G. & Ueta, T. [1999] "Yet another chaotic attractor," *Int. J. Bifurcation and Chaos* **9**, 1465–1466.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmos. Sci.* **20**, 130–141.

Lü, J. H. & Chen, G. [2002] "A new chaotic attractor coined," *Int. J. Bifurcation and Chaos* **12**, 659–661.

Lü, J. H., Chen, G., Cheng, D. & Čelikovský, S. [2002] "Bridge the gap between the Lorenz system and the Chen system," *Int. J. Bifurcation and Chaos* **12**, 2917–2926.

Qi, G., Chen, G., Du, S., Chen, Z. & Yuan, Z. [2005] "Analysis of a new chaotic system," *Physica A* **352**, 295–308.

Sparrow, C. [1982] *The Lorenz Equations: Bifurcation, Chaos, and Strange Attractor* (Springer-Verlag, NY).

Sprott, J. C. [1993] "Automatic generation of strange attractors," *Comput. Graph.* **17**, 325–332.

Sprott, J. C. [1994] "Some simple chaotic flows," *Phys. Rev. E* **50**, 647–650.

Sprott, J. C. [1997] "Simplest dissipative chaotic flow," *Phys. Lett. A* **228**, 271–274.

Sprott, J. C. & Linz, S. J. [2000] "Algebraically simple chaotic flows," *J. Chaos Th. Appl.* **5**, 3–22.

Ueta, T. & Chen, G. [2000] "Bifurcation analysis of Chen's equation," *Int. J. Bifurcation and Chaos* **10**, 1917–1931.

Vaneček, A. & Čelikovský, S. [1996] *Control Systems: From Linear Analysis to Synthesis of Chaos* (Prentice-Hall, London).

Yang, Q. & Chen, G. [2008] "A chaotic system with one saddle and two stable node-foci," *Int. J. Bifurcation and Chaos* **18**, 1393–1414.

Yang, Q., Chen, G. & Huang, K. [2007] "Chaotic attractors of the conjugate Lorenz-type system," *Int. J. Bifurcation and Chaos* **17**, 3929–3949.

Yang, Q., Chen, G. & Zhou, T. [2006] "A unified Lorenz-type system and its canonical form," *Int. J. Bifurcation and Chaos* **16**, 2855–2871.

Zhou, T., Chen, G. & Yang, Q. [2004a] "Constructing a new chaotic system based on the Šhi'lnikov criterion," *Chaos Solit. Fract.* **19**, 985–993.

Zhou, T., Tang, Y. & Chen, G. [2003] "Complex dynamical behaviors of the chaotic Chen's system," *Int. J. Bifurcation and Chaos* **13**, 2561–2574.

Zhou, T., Tang, Y. & Chen, G. [2004b] "Chen's attractor exists," *Int. J. Bifurcation and Chaos* **14**, 3167–3177.

Zhou, T., Tang, Y. & Chen, G. [2006] "Classification of chaos in 3-D autonomous quadratic systems-I. Basic framework and methods," *Int. J. Bifurcation and Chaos* **16**, 2459–2479.