International Journal of Bifurcation and Chaos, Vol. 23, No. 4 (2013) 1330011 (20 pages)

DOI: 10.1142/S0218127413300115

A GALLERY OF LORENZ-LIKE AND CHEN-LIKE ATTRACTORS

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Received February 11, 2013

In this article, three-dimensional autonomous chaotic systems with two quadratic terms, similar to the Lorenz system in their algebraic forms, are studied. An attractor with two clearly distinguishable scrolls similar to the Lorenz attractor is referred to as a Lorenz-like attractor, while an attractor with more intertwine between the two scrolls similar to the Chen attractor is referred to as a Chen-like attractor. A gallery of Lorenz-like attractors and Chen-like attractors are presented. For several different families of such systems, through tuning only one real parameter gradually, each of them can generate a spectrum of chaotic attractors continuously changing from a Lorenz-like attractor to a Chen-like attractor. Some intrinsic relationships between the Lorenz system and the Chen system are revealed and discussed. Some common patterns of the Lorenz-like and Chen-like attractors are found and analyzed, which suggest that the instability of the two saddle-foci of such a system somehow determines the shape of its chaotic attractor. These interesting observations on the general dynamic patterns hopefully could shed some light for a better understanding of the intrinsic relationships between the algebraic structures and the geometric attractors of these kinds of chaotic systems.

Keywords: Chaotic attractor; Lorenz system; Chen system; generalized Lorenz canonical form.

1. Introduction

The now-classic Lorenz system [Lorenz, 1963; Sparrow, 1982] is described by

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy, \end{cases} \tag{1}$$

which is chaotic when $\sigma = 10$, $r = 28$, $b = \frac{8}{3}$.

Ever since its discovery in 1963, the Lorenz system has been a paradigm of chaos; the Lorenz attractor, which vividly resembles a butterfly's wings, has become an emblem of chaos; Edward N. Lorenz himself, has been marked by the history as an icon of chaos.

System (1) is a three-dimensional (3D) autonomous system with only two quadratic terms in its nonlinearity, which is very simple in the algebraic structure and yet is fairly complex in dynamical behaviors. The observation of these two seemingly contradictory aspects of the Lorenz system thereby triggered a great deal of interest from the scientific community to seek closely-related Lorenz-like systems, by different motivations and from various perspectives. Along the same line of curiosity and interest, this article considers the following issues:

(i) Concerning the algebraic structure of the chaotic Lorenz system, are there similar or even simpler systems of ordinary differential

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equations capable of producing two-wing chaotic attractors?

(ii) Concerning the geometry of the Lorenz attractor, are there other symmetrical attractors in different shapes generated by systems of similar structures?

(iii) After all, what is the intrinsic relationship between the algebraic structure and the geometric dynamics of such a chaotic system?

Regarding question (i), there has been some progress in the last decade. In 1999, from an engineering feedback anti-control approach, Chen coined a new chaotic system [Chen & Ueta, 1999; Ueta & Chen, 2000], lately referred to as the Chen system by others, described as

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = -bz + xy, \end{cases} \tag{2}$$

which is chaotic when $a = 35$, $b = 3$, $c = 28$.

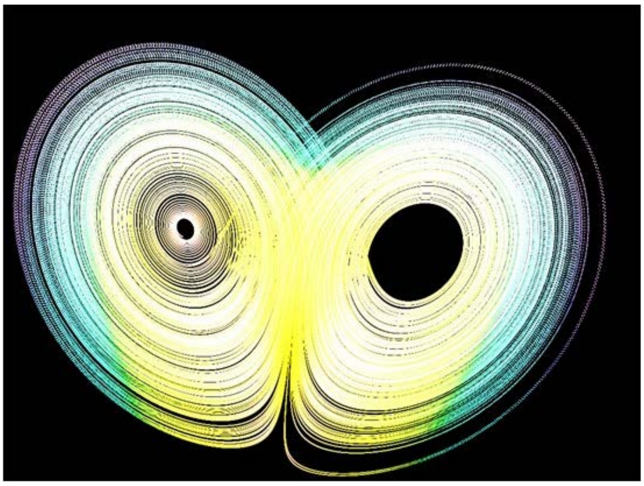
The Lorenz attractor and Chen attractor are shown in Fig. 1. In the following, an attractor with two clearly distinguishable scrolls is referred to as a *Lorenz-like attractor*, while an attractor with more intertwine between the two scrolls as the *Chen-like attractor*.

The significance of the Chen system lies in its following features comparable to but different from the Lorenz system:

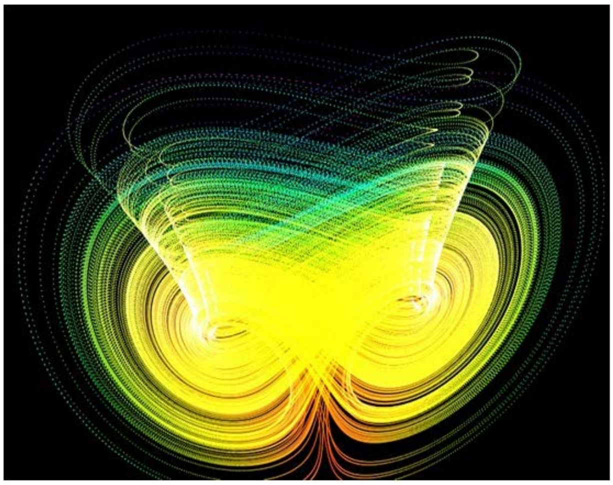
- it is a 3D autonomous system with only two quadratic terms in its nonlinearity;
- it has three equilibria: one saddle and two saddle-foci;
- its attractor is globally bounded [Barboza & Chen, 2011];
- it is nonequivalent to the Lorenz system [Hou *et al.*, 2010];
- it is *dual* to the Lorenz system in the sense of Čelikovský and Vaneček [1994];
- it has more sophisticated dynamical behaviors than the Lorenz system [Ueta & Chen, 2000; Zhou *et al.*, 2003].

More importantly, in between the Lorenz system and the Chen system, there are infinitely many genetically-related chaotic systems, all of which constitute a one-parameter family of generalized Lorenz systems [Vaneček & Čelikovský, 1996; Čelikovský & Chen, 2002a, 2002b, 2005].

Other than the early-found Rössler system [Rössler, 1976] and the lately-found Chen system, which has similar algebraic structure to the Lorenz system (i.e. both are 3D autonomous systems with two same quadratic terms), and some other Lorenz like systems [Zhou *et al.*, 2003; Zhou *et al.*, 2004a; Zhou *et al.*, 2004b; Zhou *et al.*, 2006; Yang *et al.*, 2006; Yang *et al.*, 2007; Yang & Chen, 2008; Qi *et al.*, 2005], it had also been found that there exist some other similar chaotic systems that have even simpler algebraic structures [Sprott, 1993, 1994, 1997; Sprott & Linz, 2000], where through a systematic search of general 3D autonomous ordinary differential equations with one or two quadratic terms, near 20 simple chaotic systems were uncovered, which have five terms with two quadratic or



(i)



(ii)

Fig. 1. Lorenz attractor and Chen attractor.

six terms with only one quadratic term in the system equations.

Regarding question (ii), there has not been much progress in the past. Although it was found that there are systems with structures similar to the Lorenz system, which can generate symmetrical attractors in different shapes, such as those mentioned above, the mathematical principles and the underlying mechanisms are far from being understood.

Regarding question (iii), recently there has not been much progress either. Nevertheless, from among the above-referred chaotic systems, it has been observed that some systems possess different algebraic structures but have similar dynamical behaviors while some systems possess similar algebraic structures but have different dynamical behaviors. These observations reveal the complexity of the issue in question — the intrinsic relationship between the structure and the dynamics of a chaotic system. This paper attempts to provide some partial answers to the question, by showing a class of Lorenz-like and Chen-like chaotic systems along with a gallery of their chaotic attractors. It hopefully can shed some light for a better understanding of the intrinsic relationships between the algebraic structures and the geometric attractors of these kinds of chaotic systems.

It is noted here that several chaotic systems to be discussed below are newly coined, which were briefly reported without detailed analysis in a recent overview published in a Chinese journal [Wang & Chen, 2012b].

2. Symmetry

Although the relationship between the system algebraic structure and chaotic dynamics seems quite subtle, one thing that can be surely predicted from the algebraic equations is the geometric symmetry of the system chaotic attractor.

The symmetry of the algebraic equations determine the symmetry of its geometric dynamics. In fact, symmetry plays an important role in generating chaos, which implicitly determines the possible shape of a resulting attractor.

For example, both the Lorenz system and the Chen system have the z-axis rotational symmetry, so they both generate a two-scroll butterfly-shaped symmetrical attractor. However, if one adds a constant control term to the second equation of the Chen system and gradually tunes the values of this constant term, then one will find that the Chen attractor gradually loses its original symmetry and finally becomes an unsymmetrical one-scroll attractor, as reported in [Wang & Chen, 2012a].

The above example hints that if one wants to generate symmetrical attractors from a 3D autonomous system with some quadratic term(s) different from that of the Lorenz system, then it needs to preserve the z-axis rotational symmetry. The z-axis rotational symmetry property requires that the system algebraic equations should remain the same when (x, y, z) is transformed to $(-x, -y, z)$. This gives a restrictive condition to the algebraic structure of a system, which one intends to construct or to find, hoping to obtain a new system with a similar structure but different dynamics.

For example, if one performs the transform $(x, y, z) \mapsto (-x, -y, z)$ to a Lorenz-like system with the second equation as the following:

$$\dot{y} = a_{21}x + a_{22}y + a_{23}z + pxz + qxy,$$

then one obtains

$$-\dot{y} = -a_{21}x - a_{22}y + a_{23}z - pxz + qxy,$$

then by comparing it with the original equation, one can see that a_{23} and q must be 0.

There are in total six possible quadratic nonlinear terms: xy, yz, xz, x^2, y^2 and z^2 in a 3D quadratic equation. Restricted by the z-axis rotational symmetry, the nonlinear terms in the second equation of the 3D system in interest must be either xz or yz, while in the third equation they must be either xy or z^2, or x^2, or y^2.

Based on the above observations, to maintain the z-axis rotational symmetry, the most general form of a 3D autonomous system with only linear and quadratic terms seems to be the following:

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y + m_1xz + m_2yz \\ \dot{z} = a_{33}z + m_3xy + m_4x^2 + m_5y^2 + m_6z^2 + c, \end{cases} \tag{3}$$

where all coefficients are real constants.

Now, for systems with only two quadratic terms, particularly with one in the second equation and the other in the third equation, there are in total $2 \times 4 = 8$ possible cases to consider. One typical example is the following form, which has the same two quadratic terms as the Lorenz system:

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y + m_1xz \\ \dot{z} = a_{33}z + m_3xy + c. \end{cases} \tag{4}$$

Notice that this system has eight constant parameters. But this number of free parameters can be further reduced through some equivalent transformations.

In general, for two vector fields $f(x)$ and $g(x)$ in $\mathbb{R}^n$, satisfying dynamical systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ respectively, $xy \in \mathbb{R}^n$, if there exists a diffeomorphism h on $\mathbb{R}^n$ such that

$$f(x) = M^{-1}(x)g(h(x)),$$

where $M(x)$ is the Jacobian of h at the point x, then the two dynamical systems are said to be *smoothly equivalent* [Kuznetsov, 1998].

Now, a simple scale transform $(x, y, z) \mapsto (px, qy, rz)$ on (4) yields the following general form:

$$\begin{cases} \dot{x} = a_{11}x + \dfrac{a_{12}q}{p}y \\ \dot{y} = \dfrac{a_{21}p}{q}x + a_{22}y + \dfrac{m_1 pr}{q}xz \\ \dot{z} = a_{33}z + \dfrac{m_3 pq}{r}xy + \dfrac{c}{r}. \end{cases} \tag{5}$$

If $m_1 m_3 > 0$, then one can set $\frac{m_1 pr}{q} = \frac{m_3 pq}{r} = 1$ by letting $p = \frac{1}{\sqrt{m_1 m_3}}$, $q = \sqrt{m_1}$ and $r = \sqrt{m_3}$. This means that the parameters of the two quadratic terms can both be fixed to be 1.

If $m_1 m_3 < 0$, then one can set $\frac{m_1 pr}{q} = -1$ and $\frac{m_3 pq}{r} = 1$ by letting $p = \frac{1}{\sqrt{-m_1 m_3}}$, $q = \sqrt{|m_1|}$ and $r = \sqrt{|m_3|}$. This means the parameters of the two quadratic terms can be fixed to 1 and -1, respectively.

Thus, by a simple linear transformation, one can always set the parameters of the two quadratic terms to be 1 or -1, thereby reducing the parameter space so as to obtain the following simpler canonical form:

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y \pm xz \\ \dot{z} = a_{33}z + xy + c. \end{cases} \tag{6}$$

For a system with any other combination of two quadratic terms, one can reformulate it into a similar form. But for a system with more than two quadratic terms, there does not exist such simplification, because the other parameters are likely not independent, therefore a simplified form may contain redundancy.

In the following, several general families of Lorenz-like systems formulated in the above canonical form will be studied.

Similar to the aforementioned generalized Lorenz systems, each of the following families of chaotic systems has one and only one tunable real parameter. As the parameter is gradually varied, a sequence of chaotic attractors can be generated from the same system, in a continuous manner, with very rich and complicated dynamics.

3. Lorenz and Chen Attractors

Recall the so-called *unified chaotic system*, which encompasses both the Lorenz system and the Chen system [Lü & Chen, 2002; Lü *et al.*, 2002]. This unified chaotic system is by nature a convex combination of the two systems, and is described by

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x) \\ \dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y \\ \dot{z} = -\dfrac{\alpha + 8}{3}z + xy. \end{cases} \tag{7}$$

When $\alpha = 0$, it is the Lorenz system while with $\alpha = 1$, it is the Chen system, and moreover for any $\alpha \in (0, 1)$ the system remains to be chaotic.

A detailed stability analysis of the unified chaotic system (7) is summarized in Table 1.

4. Sprott B System: Algebraically the Simplest

The Sprott B and Sprott C systems [Sprott, 1994] with only five terms, in which two are quadratic, are among the algebraically simplest systems that could generate Chen-like attractors.

The Sprott C system will be studied later below. The Sprott B system [Sprott, 1994] is discussed here, which can be transformed to the following conical form:

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -xz \\ \dot{z} = xy + b. \end{cases} \tag{8}$$

It is clear that this system has the same nonlinear part as the Lorenz and the Chen systems, but has a much simpler linear part. Some of its typical chaotic attractors are shown in Fig. 2. Note, however, that this system has only two saddle-foci with the same eigenvalues, which is different from both the Lorenz and Chen systems. A detailed stability analysis of the system (8) is given in Table 1.

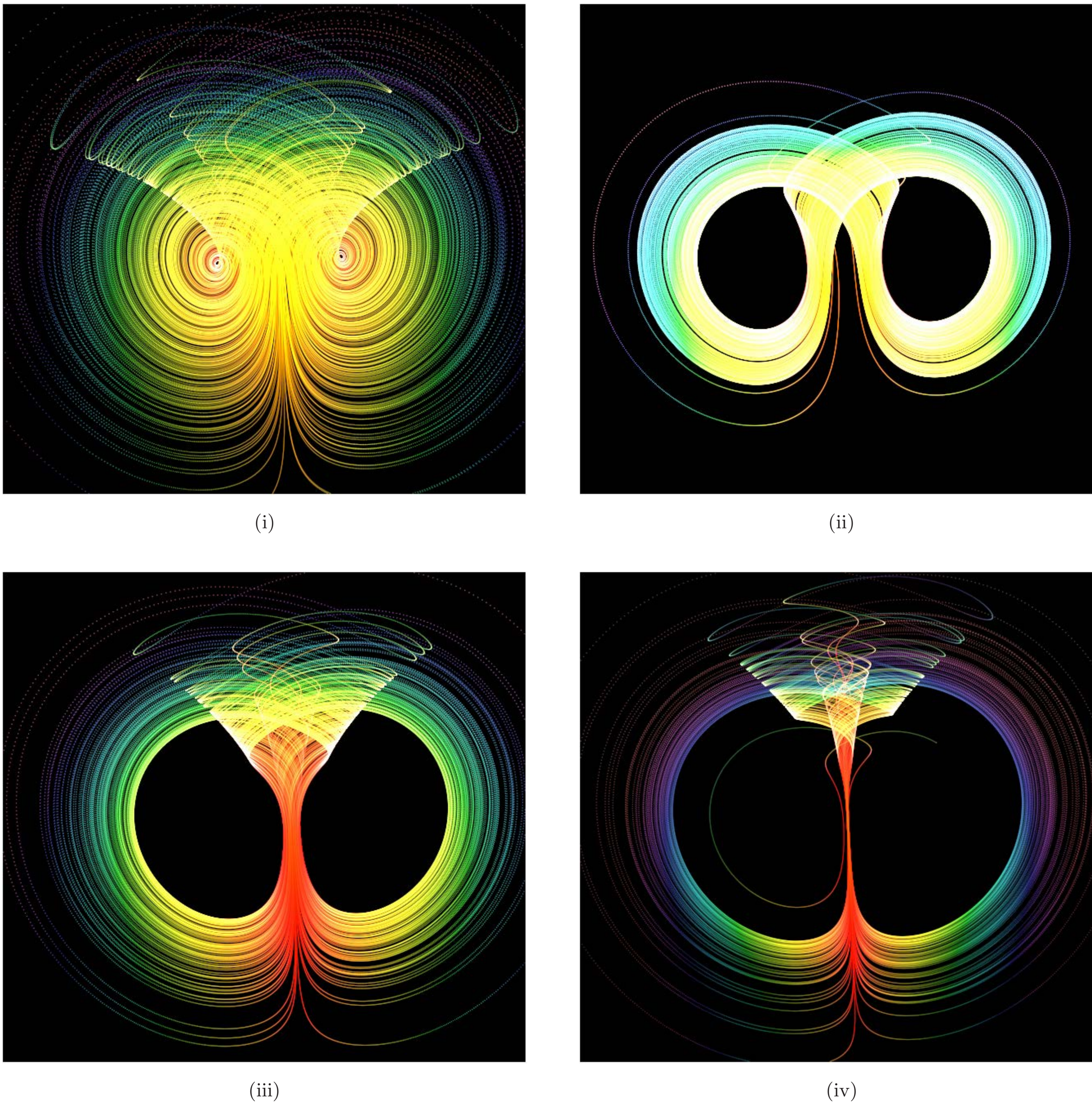


Fig. 2. Attractors of the Sprott B system: (i) $b = -1$, (ii) $b = -0.35$, (iii) $b = -0.2$ and (iv) $b = -0.13$.

5. Family-c: Chaos with Stable Node-Foci

The following is another family of Lorenz-like systems, introduced in [Wang & Chen, 2012a]:

$$\begin{cases} \dot{x} = -x - y \\ \dot{y} = -x + cy - xz \\ \dot{z} = -0.1z + xy, \end{cases} \tag{9}$$

where c is a real parameter.

The differences between system (9) and the unified chaotic system (7) are quite subtle. For system (9), the real part of its complex conjugate eigenvalues is precisely zero when $c = 0.05$. This is a critical point at which the stability of the equilibria is changed from stable to unstable.

An interesting phenomenon is that the chaotic attractor does not disappear right after the two unstable saddle-foci became stable node-foci. Instead, there still exists a globally chaotic attractor

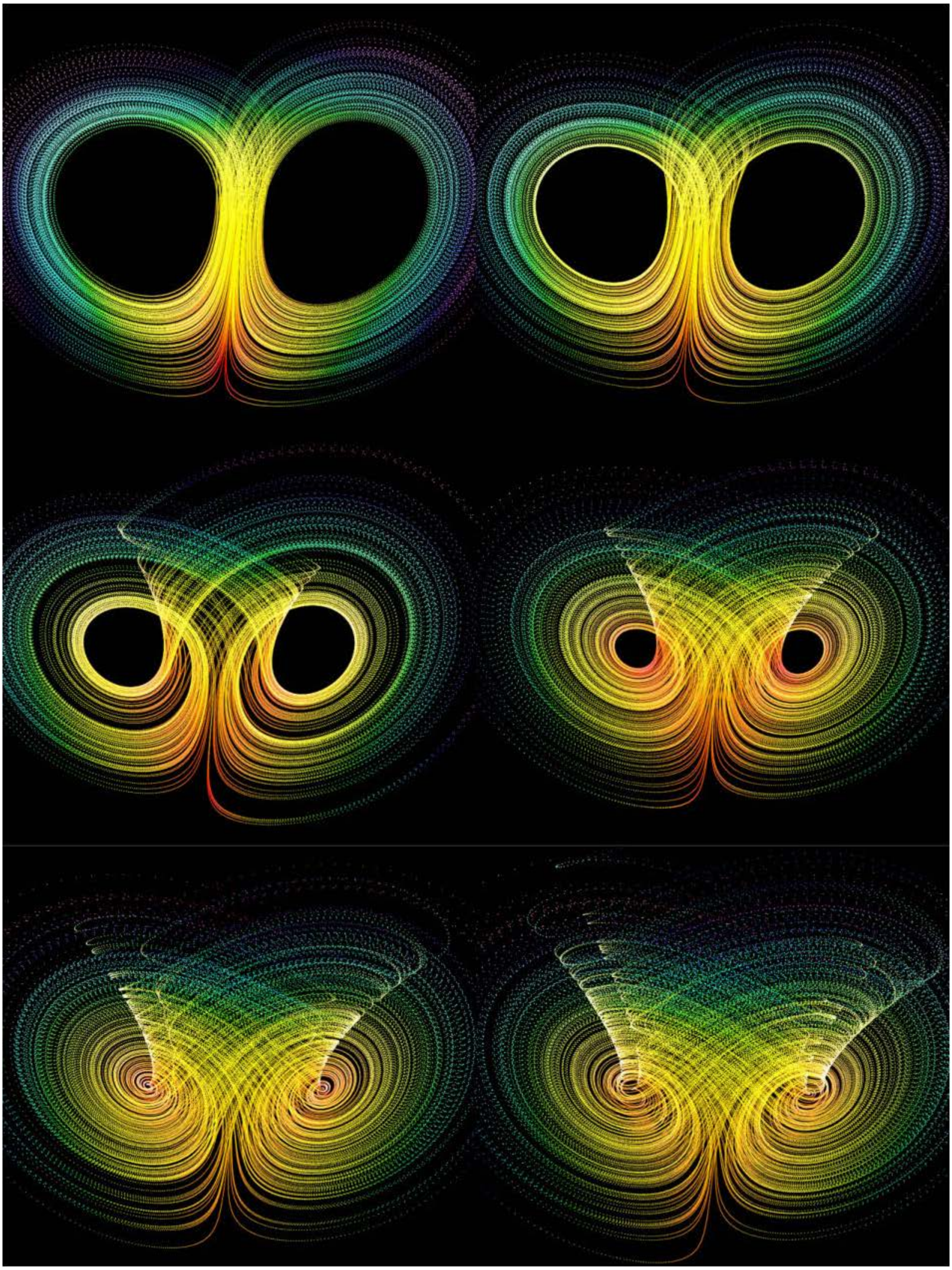
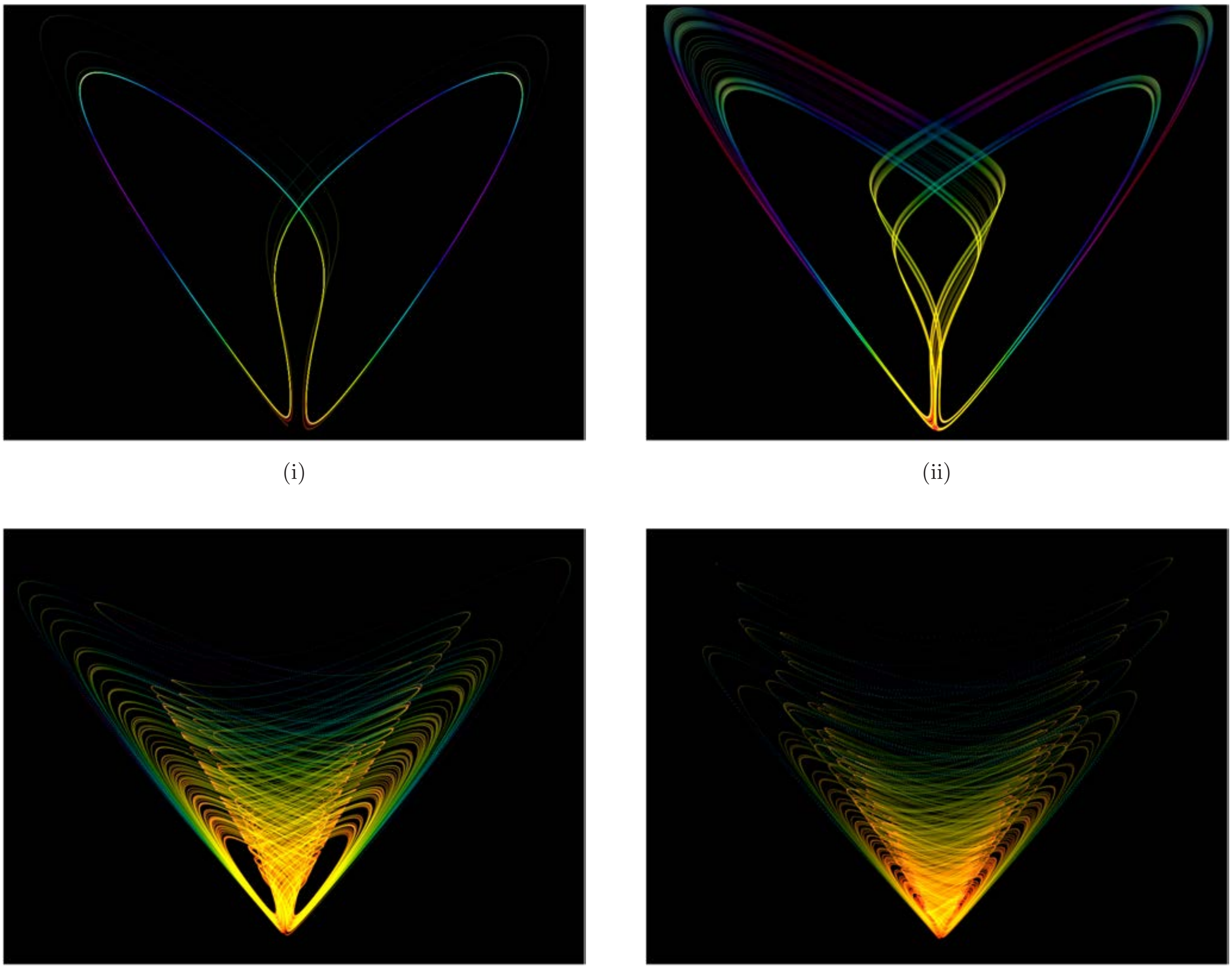


Fig. 3. Lorenz–Chen attractor transition in the family-c of chaotic systems.



(i) (ii)

(iii) (iv)

Fig. 4. Attractors of family-d of chaotic systems: (i) $d = 0.25$, (ii) $d = 0.28$, (iii) $d = 0.45$ and (iv) $d = 0.7$.

along with locally convergent behaviors near the two stable node-foci. Since the parameter c may take values on either sides of $c = 0.05$, both can generate chaos, the new system (9) has richer dynamics and appears to be more interesting than the unified system (7).

6. Family-d: Slim Chen-Like Attractor

All the above systems have the same nonlinear terms as the Lorenz system. The following is another interesting family of chaotic systems, family-d, described by

$$\begin{cases} \dot{x} = dx - 0.1y \\ \dot{y} = x - y + xz \\ \dot{z} = xy - 0.2z - 0.1, \end{cases} \tag{10}$$

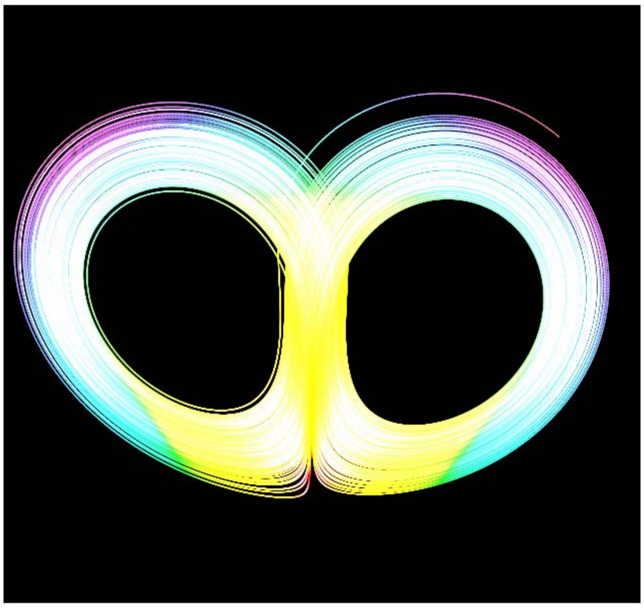
where d is a real parameter, which can generate a sequence of slim chaotic Chen-like attractors, as shown in Fig. 4.

7. Family-e with Quadratic Terms $-xz$ and x^2

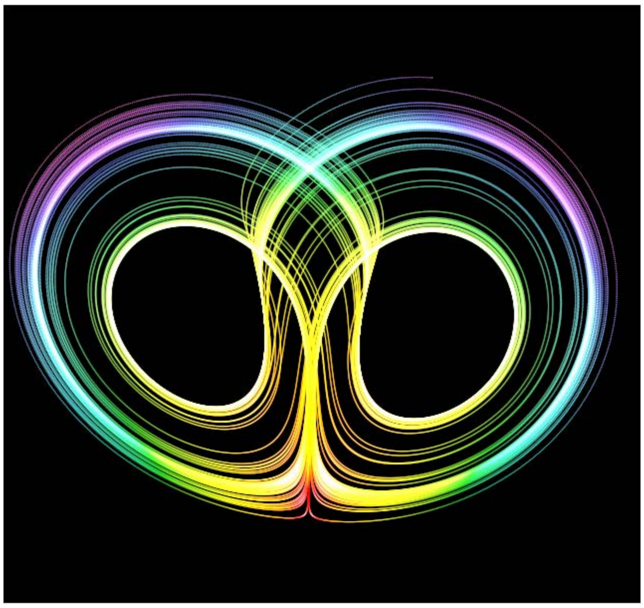
This family of chaotic systems with quadratic terms $-xz$ and x^2 is described by

$$\begin{cases} \dot{x} = -ex - y \\ \dot{y} = -x + xz \\ \dot{z} = -0.2z + x^2, \end{cases} \tag{11}$$

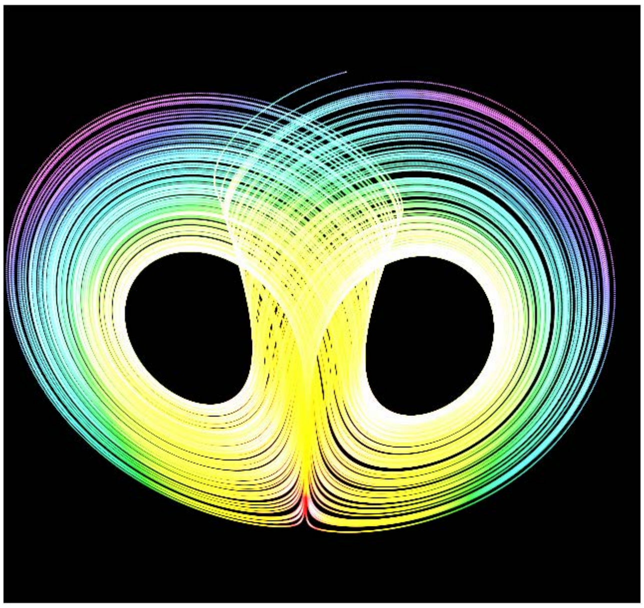
where e is a real parameter. This system can generate both Lorenz-like and Chen-like chaotic attractors, as shown in Fig. 5. Although the nonlinear terms are quite different from that of the Lorenz



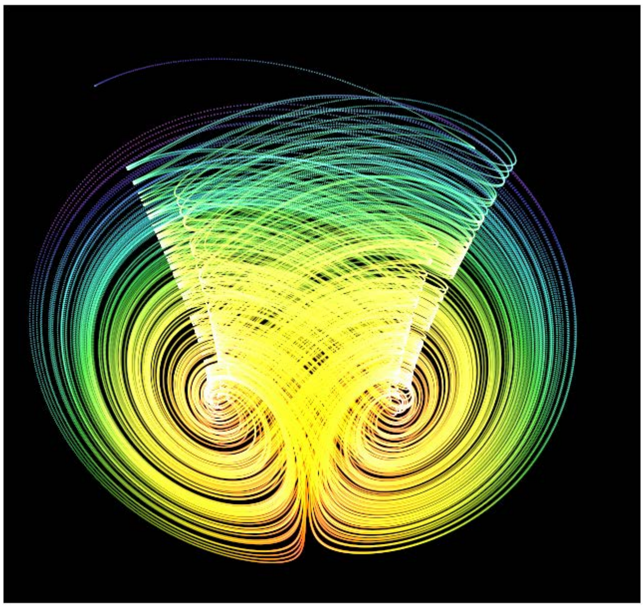
(i)



(ii)



(iii)



(iv)

Fig. 5. Family-e of Chen-like chaotic systems: (i) $e = 1.2$, (ii) $e = 0.9$, (iii) $e = 0.8$ and (iv) $e = 0.3$.

and Chen systems, this family of systems show the same transition pattern from Lorenz-like attractors to Chen-like attractors. Also, there could be some narrow parameter range when the unstable saddle-foci become stable node-foci on which Lorenz-like chaotic attractors could still be generated.

8. Lorenz-Like System with Quadratic Terms $-xz$ and x^2

The family of Lorenz-like chaotic systems with quadratic terms of $-xz$ and x^2 is particularly interesting, which is discussed in this section.

8.1. *Family-f: Distorted Lorenz-like and Chen-like attractor*

This family of Lorenz-like chaotic systems is described by

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = x + fy - xz \\ \dot{z} = -0.4z + x^2 - 1, \end{cases} \tag{12}$$

where f is a real parameter. This system can generate some distorted Lorenz-like and Chen-like chaotic attractors, as shown in Fig. 6.

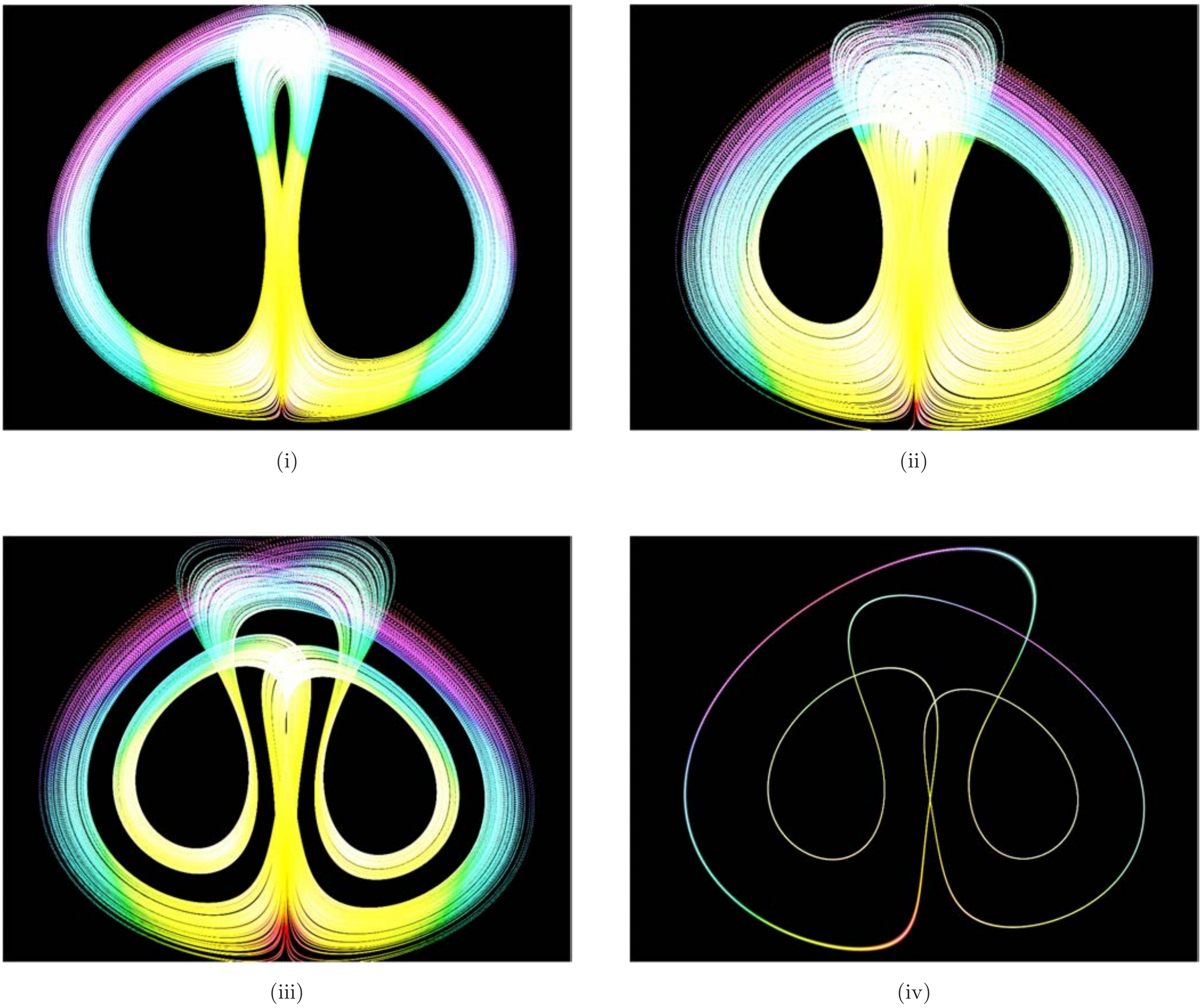
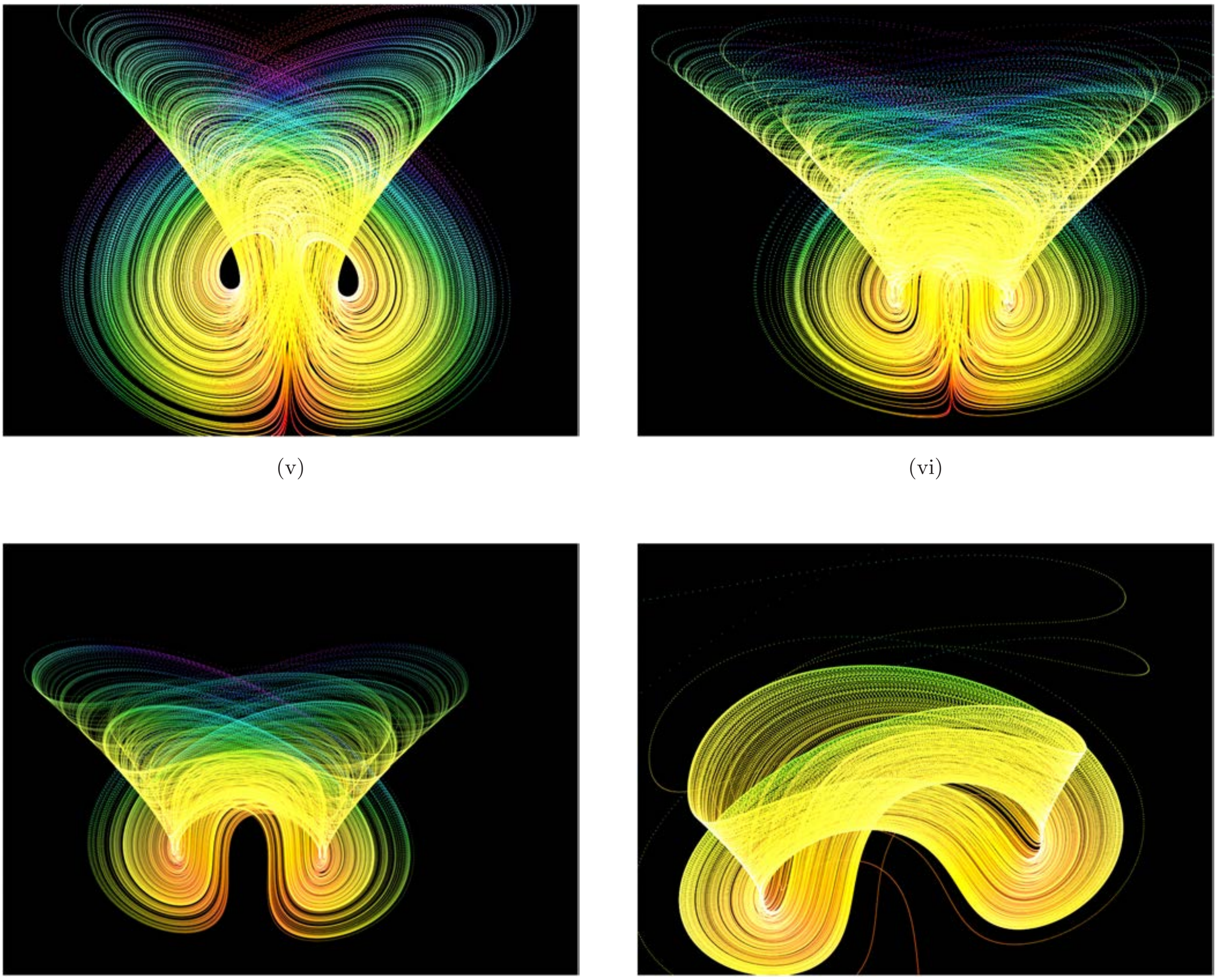


Fig. 6. Family-f of Lorenz-like chaotic systems: (i) $f = -1$, (ii) $f = -0.8$, (iii) $f = -0.7$, (iv) $f = -0.6$, (v) $f = -0.3$, (vi) $f = 0.2$, (vii) $f = 0.35$ and (viii) $f = 0.37$.



(v) (vi) (vii) (viii)

Fig. 6. (*Continued*)

8.2. *Family-g: Super Chen-like attractor*

This family of Lorenz-like chaotic systems is described by

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = x + gy - xz \\ \dot{z} = -0.1z + x^2 - 1, \end{cases} \tag{13}$$

where g is a real parameter. This system can generate Super Chen-like chaotic attractors, as shown in Fig. 7. The twist between the two scrolls seems even more drastic than the Chen-like attractors.

9. Family of Chaotic Systems with Quadratic Terms xz and y^2

Another family of chaotic systems with quadratic terms of xz and y^2 is equally interesting, which is referred to as family-h and is discussed in this section.

This family of chaotic systems is described by

$$\begin{cases} \dot{x} = -x - y \\ \dot{y} = -x + 0.9y + xz \\ \dot{z} = -hz + y^2 - 1, \end{cases} \tag{14}$$

where h is a real parameter. This system can also generate chaotic attractors, from Lorenz-like to Chen-like attractors, as shown in Fig. 8.

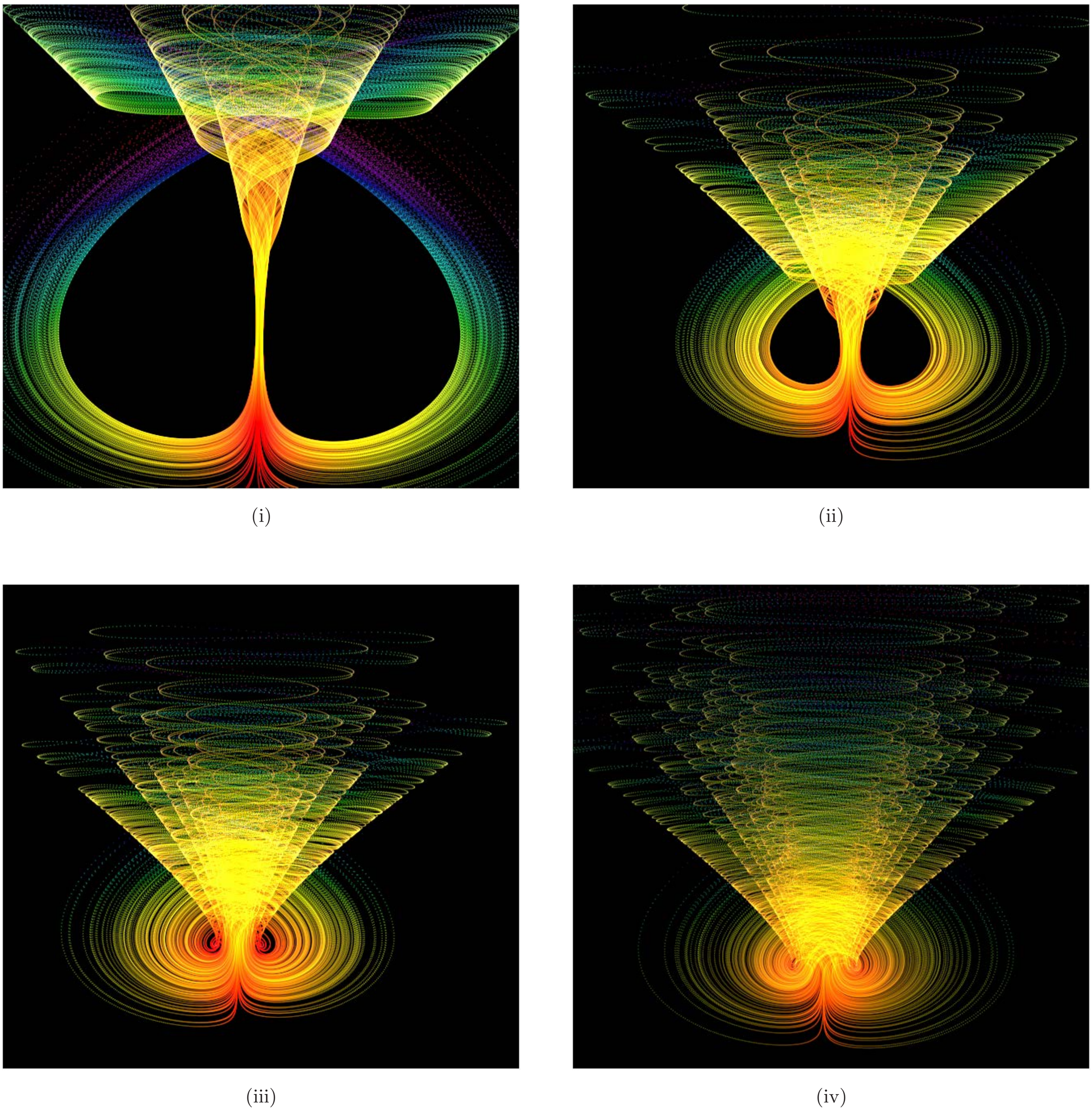
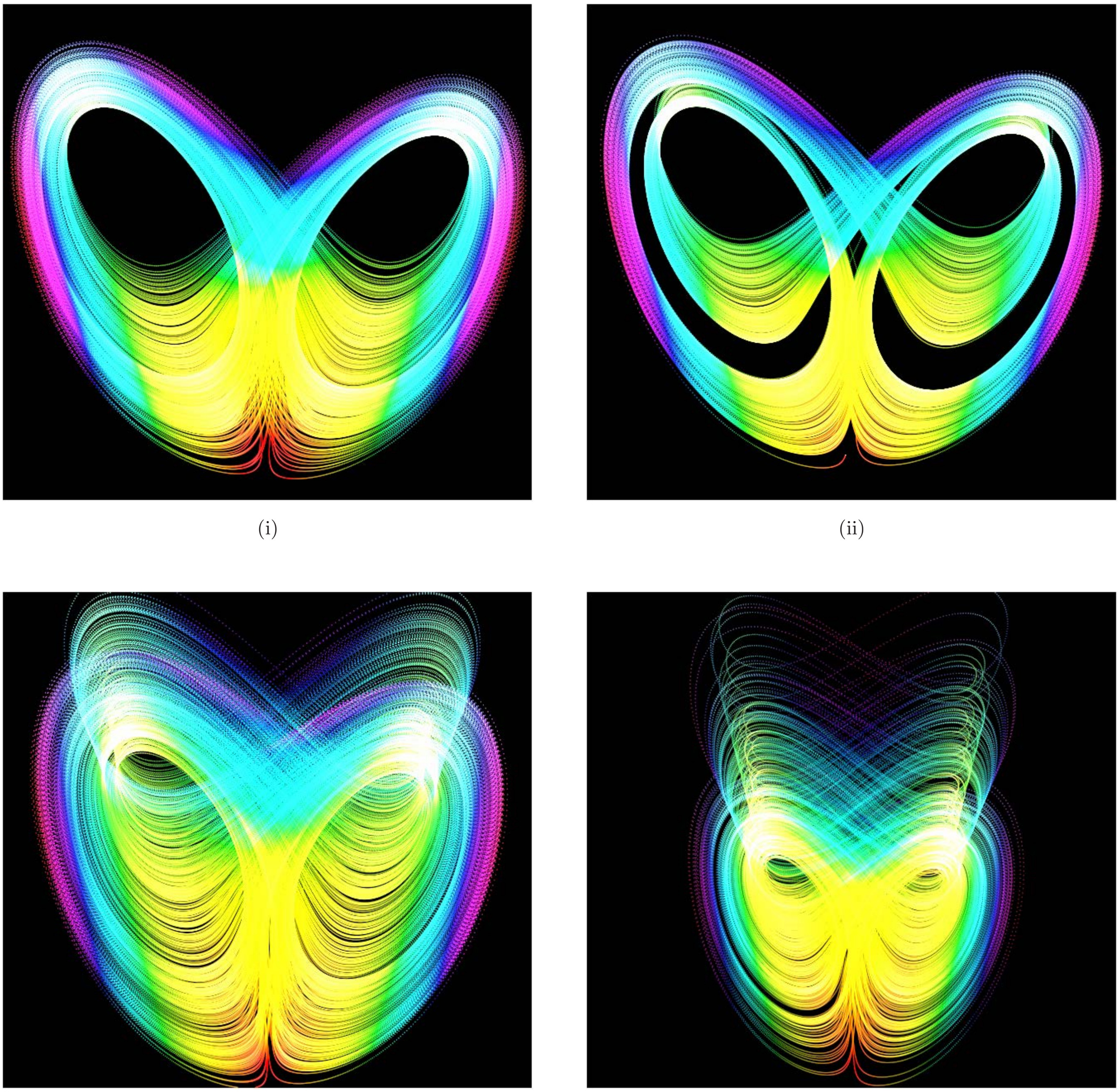


Fig. 7. Family-g of Lorenz-like chaotic systems: (i) $g = -0.4$, (ii) $g = 0$, (iii) $g = 0.2$ and (iv) $g = 0.5$.



(i) (ii) (iii) (iv)

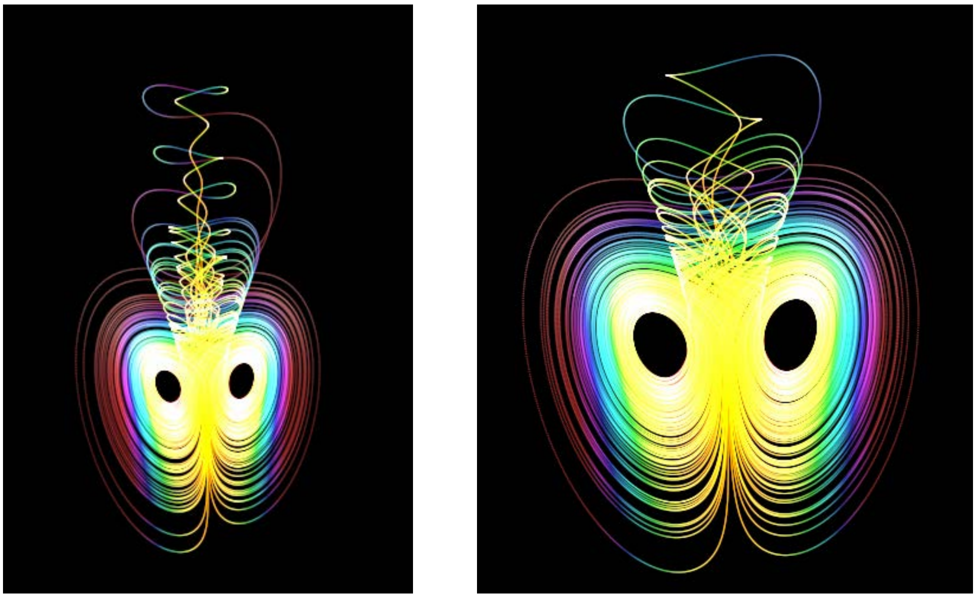
Fig. 8. Family-g of Lorenz-like chaotic systems: (i) $h = 0.8$, (ii) $h = 0.75$, (iii) $h = 0.6$ and (iv) $h = 0.45$.

10. Family of Lorenz-Like Chaotic Systems with Quadratic Terms $-xz$ and y^2

Yet another family of Lorenz-like chaotic systems with quadratic terms of $-xz$ and y^2 is similarly interesting, which is discussed in this section.

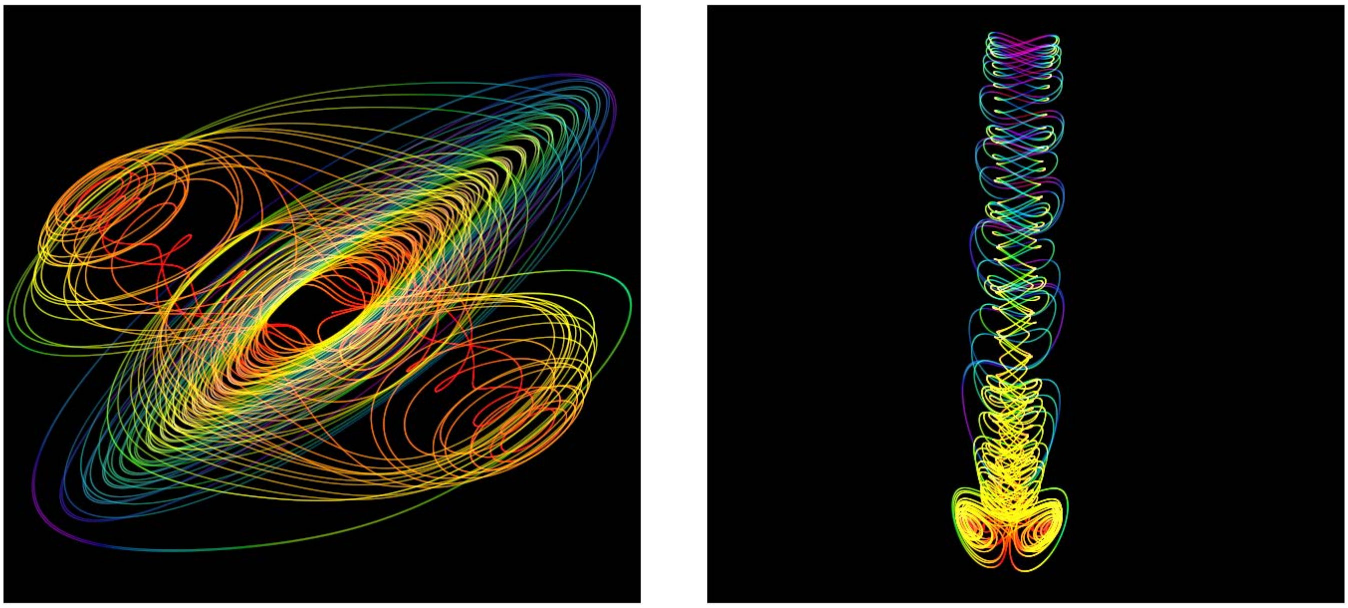
10.1. *Sprott C system*

As mentioned above, the Sprott B and Sprott C systems, with only five terms, in which two are quadratic, are among the algebraically simplest systems that could generate Chen-like attractors.



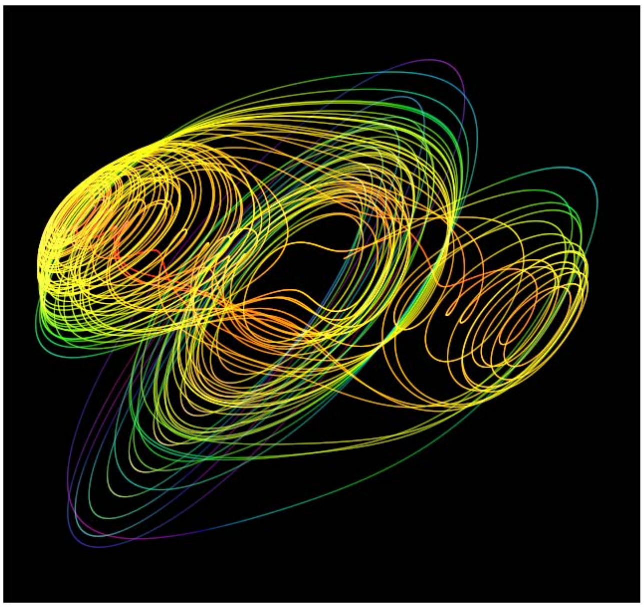
(i) (ii)

Fig. 9. The Sprott C system: (i) $i = 1$ and (ii) $i = 0.8$.

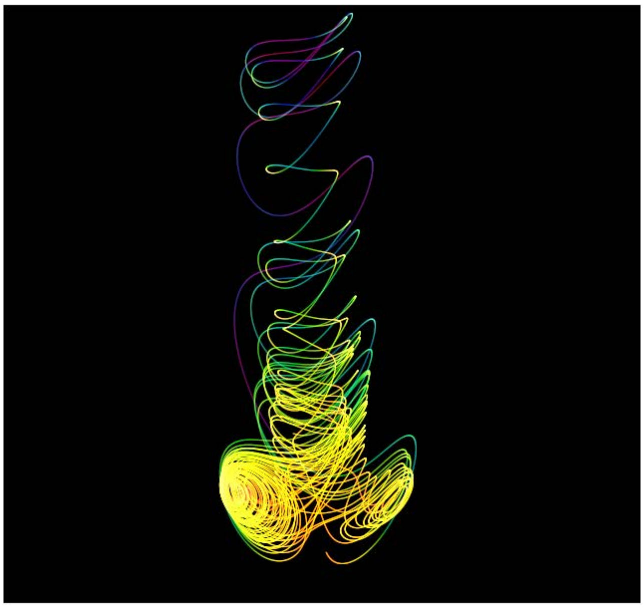


(i) (ii)

Fig. 10. Family-j of chaotic systems: (i, ii) $j = 0.1$, (iii, iv) $j = 0.15$ and (v, vi) $j = 0.2$.



(iii)



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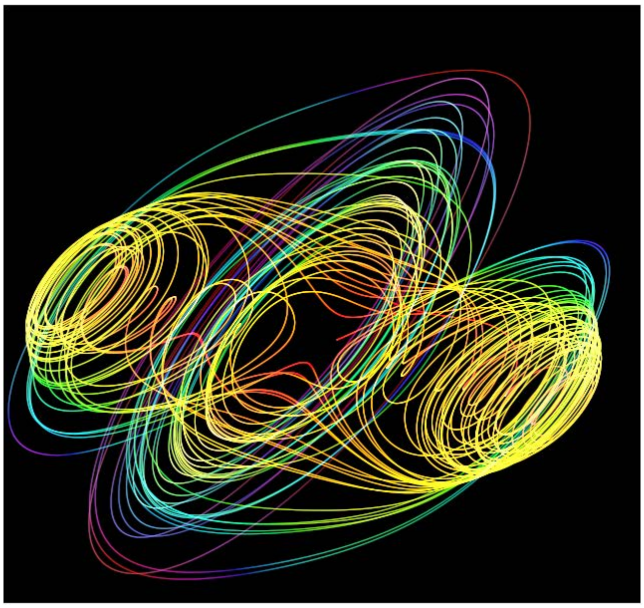
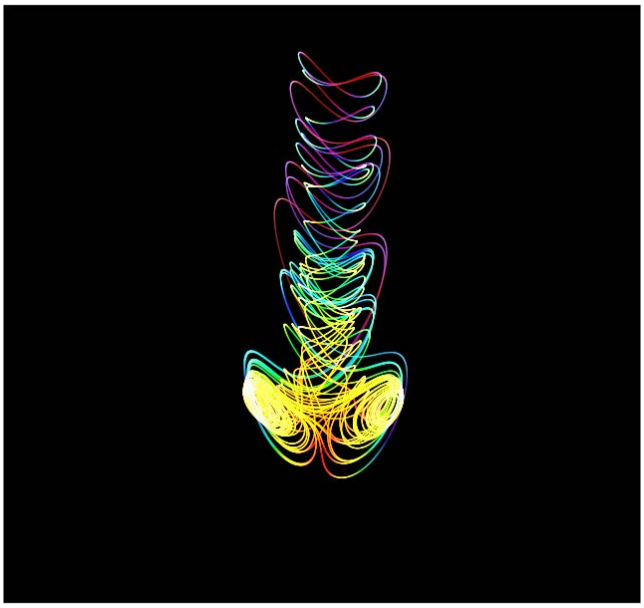


(v)



(vi)

Fig. 10. (*Continued*)

The Sprott C system is described by [Sprott, 1994]

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -xz \\ \dot{z} = y^2 - i, \end{cases} \tag{15}$$

where i is a real parameter. This system can generate Lorenz-like chaotic attractors, as shown in Fig. 9.

10.2. *Family-j: Strange Chen-like attractor*

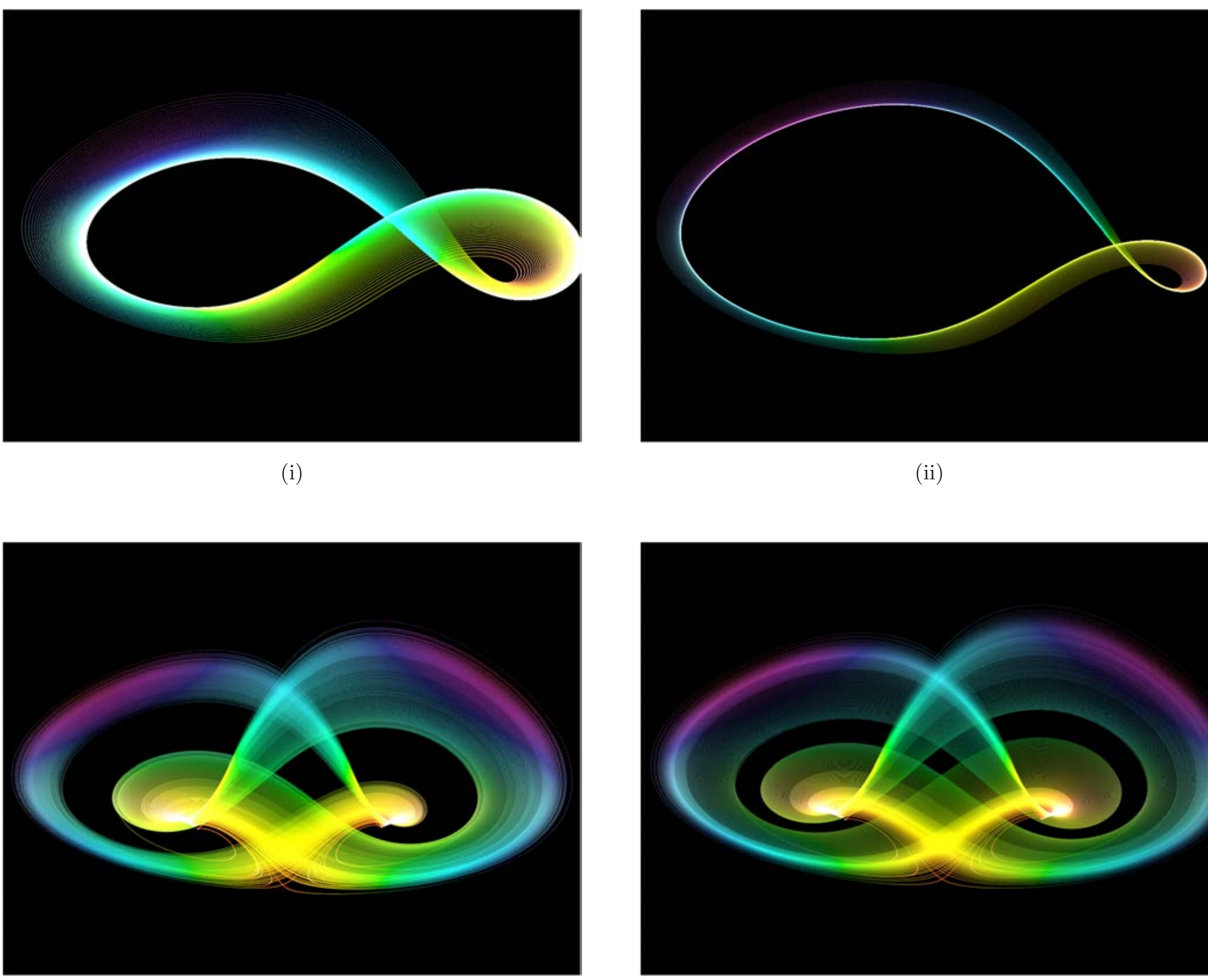
This family of chaotic systems is described by

$$\begin{cases} \dot{x} = -0.4x + y \\ \dot{y} = x + 0.3y - xz \\ \dot{z} = -jz + y^2 - 1, \end{cases} \tag{16}$$

where j is a real parameter, which can generate some "strange" Chen-like chaotic attractors, as shown in Fig. 10.

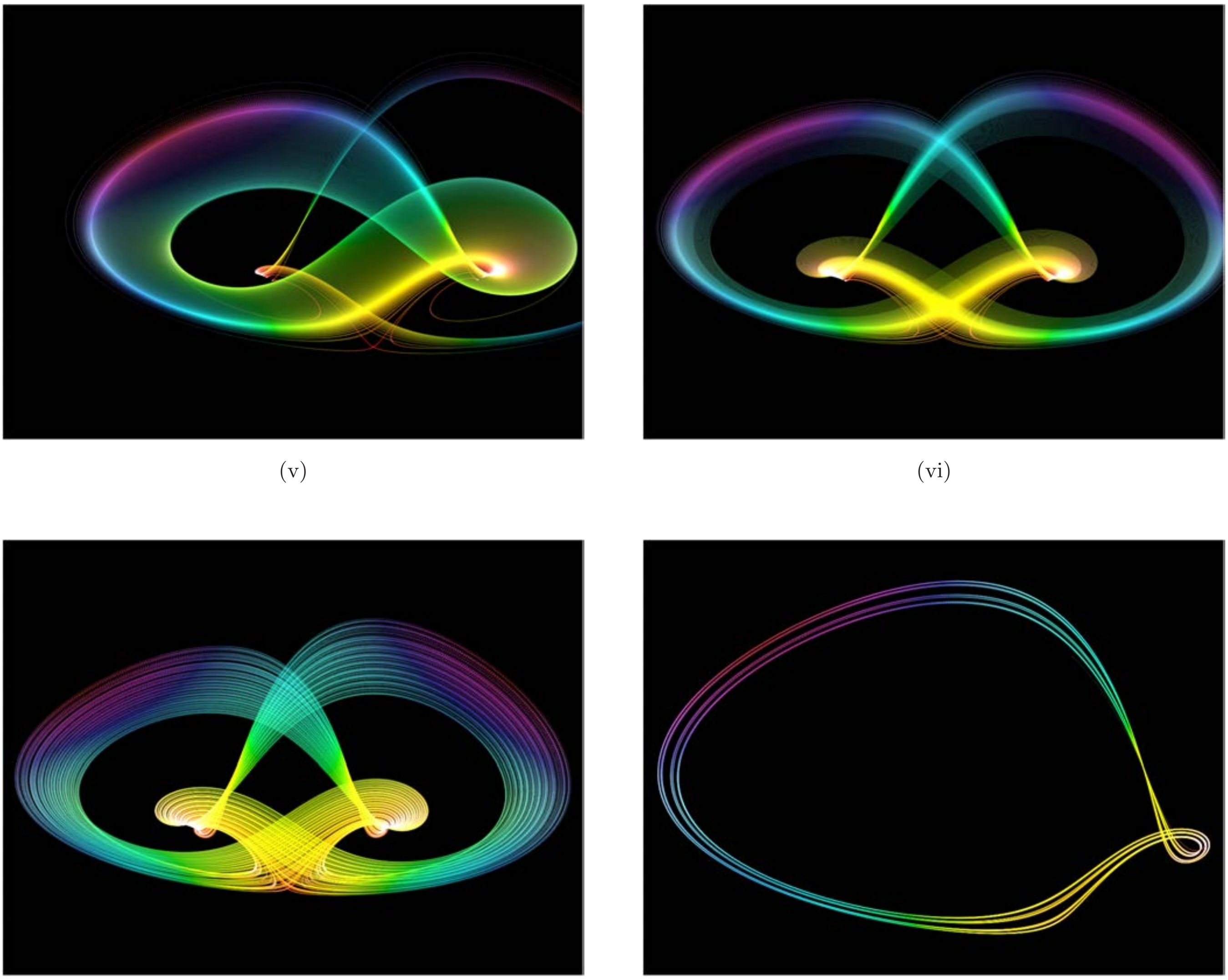
11. Family-k: Coexistence of Two Symmetrical Attractors

Constrained by the algebraic symmetry of the system equations, all the above systems generate symmetrical two-scroll attractors. But this is not always the case, because this statement is based on the



(i) (ii) (iii) (iv)

Fig. 11. Family-k of chaotic systems with two coexisting attractors: (i) $k = 0.78$, (ii) $k = 0.786$, (iii) $k = 0.7892$, (iv) $k = 0.7893$, (v) $k = 0.7896$, (vi) $k = 0.79$, (vii) $k = 0.81$ and (viii) $k = 0.82$.



(v) (vi)

(vii) (viii)

Fig. 11. (*Continued*)

postulation that the dynamic system has only one attractor. However, nonlinear systems may have multiple coexisting attractors.

The most fascinating family of chaotic systems discussed here seems to be the following, each system of which has two symmetrical attractors coexisting side by side at the same time. This family of chaotic systems is described by

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -x - 0.1y - yz \\ \dot{z} = -0.5z + x^2 - k, \end{cases} \tag{17}$$

where k is a real parameter. It can generate coexisting chaotic attractors, as shown in Fig. 11. For clarity, these figures only show one of the two coexisting attractors, so they appear losing the symmetry. Actually, there is another exact copy of each attractor, located symmetrically with each one shown in the figures.

12. Concluding Remarks and Future Research

The gallery of attractors presented in this article has demonstrated many novel Lorenz-like and Chen-like attractors generated by 3D autonomous systems with two quadratic terms that can maintain the z-axis rotational symmetry. Many of them are new and interesting in both theory and perspective. Moreover, many of them have some novel properties, therefore deserve further investigation in the future.

12.1. *Common patterns of Lorenz-like and Chen-like attractors*

Some interesting observations are summarized in the following:

12.1.1. *Transition from Lorenz-like to Chen-like attractors*

All these systems are algebraically simple, with only one tunable real parameter in their linear parts. By tuning this parameter, some systems can gradually generate a chain of chaotic attractors just like the process from Lorenz attractor to Chen attractor in the unified chaotic system [Lü *et al.*, 2002]. This common pattern is very important and the mechanism behind the transition is worthy of further investigation.

12.1.2. *Stabilities of equilibria*

The Lorenz system and the Chen system both have three equilibria: one saddle and two saddle-foci. Moreover, the two saddle-foci have the same eigenvalues: one negative real number and two complex conjugate numbers with a positive real part. An interesting pattern is due to the positive real part of the complex conjugate eigenvalues, that increases which will change the corresponding attractor from Lorenz-like to Chen-like in shape (see Tables 1 and 2 for details). Specifically, for the unified chaotic system [Lü *et al.*, 2002], the positive real part of the complex conjugate eigenvalues increases from (0.0940) (Lorenz attractor) to (4.2140) (Chen attractor). All other families of systems discussed in this paper have similar patterns. For family-g, in particular, increasing the positive real part of the complex conjugate eigenvalues will change the attractor from Chen-like to Super Chen-like in shape. This suggests that the instability of the two saddle-foci somehow determines the shape of the attractor. The larger the real part of the complex conjugate eigenvalues, the faster the orbit spirals out, hence the more Chen-like the attractor appears in shape.

12.1.3. *Prior to Lorenz-like shape: Chaotic systems with a stable equilibrium*

At the very beginning of the changing process of the Lorenz-like attractor, there could exist an amazing phenomenon that the chaotic system has only one equilibrium which is a stable node-foci. For example, in family-c, the real part of its complex conjugate eigenvalues is precisely zero when $c = 0.05$. This is a critical point at which the stability of the two equilibria is changed from stable to unstable. However, the chaotic attractor does not disappear right after the change; instead, the global chaotic attractor remains to be existent, regardless of the locally convergent behavior of the system orbits near the two stable node-foci. These phenomenon also exists in family-e, and some others.

12.1.4. *Posterior to Chen-like shape*

Changing the shape from Lorenz-like to Chen-like, the attractor becomes more and more twisted in shape and finally disappears. After changing through the end of the Chen-like shape, the attractor could become very different in appearance, such as the attractor shown in family-f (viii). These phenomenon can also be found in some other systems.

12.2. *Future research*

In this article, many attractors have been visualized from different views, so that they appear in various shapes for easy distinction. The selection of these chaotic systems are rather subjective but by no means exhaustive. It is believed that some other similar systems could also generate attractors with similar appearances. However, what has been demonstrated here suffice "to see a world in a grain of sand." Some more common patterns of the Lorenz-like and Chen-like systems could be further generated or extended from these given examples. There is still more room left for future research on the relationships between the algebraic structures and the dynamical behaviors of various chaotic systems, even for simple 3D autonomous quadratic systems.

Finally, it is noted that several chaotic systems discussed in this article are newly coined, which have been briefly reported without detailed analysis in a recent overview in a Chinese journal [Wang & Chen, 2012b]. Their analytic properties and simulated dynamics deserve further investigations.

Acknowledgment

This research was supported by the Hong Kong Research Grants Council under the GRF Grant CityU1109/12E.

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Appendix

The eigenvalues of Jacobian matrix evaluated at the three equilibria of all the chaotic system families are listed in Tables 1 and 2 for comparison.

Table 1. Equilibria and Jacobian eigenvalues of several Lorenz-like and Chen-like systems.

Systems	Equations	Equilibria	Eigenvalues
Lorenz System	$\dot{x} = 10(y - x)$	$(0, 0, 0)$	$-22.8277, -2.6667, 11.8277$
$\alpha = 0$	$\dot{y} = 28x - y - xz$		
	$\dot{z} = -\frac{8}{3}z + xy$	$(\pm 8.4853, \pm 8.4853, 27)$	$-13.8546, \mathbf{0.0940} \pm 0.1945i$

Table 1. (*Continued*)

Systems	Equations	Equilibria	Eigenvalues
Chen System $\alpha = 1$	$\dot{x} = 35(y-x)$ $\dot{y} = -7x + 28y - xz$ $\dot{z} = -3z + xy$	$(0, 0, 0)$ $(\pm 7.9373, \pm 7.9373, 21)$	$-30.8357, -3, 23.8359$ $-18.4288, \mathbf{4.2140} \pm 14.8846i$
Sprott B $b = -1$ Chen-like attractor	$\dot{x} = -x + y$ $\dot{y} = -xz$ $\dot{z} = xy - 1$	$(\pm 1, \pm 1, 0)$	$-1.3532, \mathbf{0.1766} \pm 1.2028$
Family-c $c = 0.05$ Lorenz-like attractor	$\dot{x} = -x - y$ $\dot{y} = -x + 0.05y - xz$ $\dot{z} = -0.1z + xy$	$(0, 0, 0)$ $(\pm 0.3240, \mp 0.3240, -1.05)$	$-0.1, 0.6544, -1.6044$ $-1.05, \mathbf{0} \pm 0.4472i$
Family-c $c = 0.7$ Chen-like attractor	$\dot{x} = -x - y$ $\dot{y} = -x + 0.7y - xz$ $\dot{z} = -0.1z + xy$	$(0, 0, 0)$ $(\pm 0.4123, \mp 0.4123, -1.7)$	$-0.1, 1.1624, -1.4624$ $-0.7446, \mathbf{0.1723} \pm 0.6534i$
Family-d $d = 0.28$ Lorenz-like attractor	$\dot{x} = 0.28x - 0.1y$ $\dot{y} = x - y + xz$ $\dot{z} = xy - 0.2z - 0.1$	$(0, 0, -0.5)$ $(\pm 0.4053, \pm 1.1349, 1.80)$	$-0.9597, -0.2000, 0.2397$ $-1.0270, \mathbf{0.0535} \pm 0.2945i$
Family-d $d = 0.7$ Chen-like attractor	$\dot{x} = 0.7x - 0.1y$ $\dot{y} = x - y + xz$ $\dot{z} = xy - 0.2z - 0.1$	$(0, 0, -0.5)$ $(\pm 0.4309, \pm 3.0166, 6.0)$	$-0.9701, -0.2000, 0.6701$ $-0.9332, \mathbf{0.2166} \pm 0.4813i$
Family-e $e = 0.2$ Chen-like attractor	$\dot{x} = -y - 0.2x$ $\dot{y} = -x + xz$ $\dot{z} = -0.2z + x^2$	$(0, 0, 0)$ $(\pm 0.4472, \mp 0.0894, -1)$	$-1.1050, -0.2, 0.9050$ $-0.8758, \mathbf{0.2379} \pm 0.6326i$
Family-e $e = 1$ Lorenz-like attractor	$\dot{x} = -y - x$ $\dot{y} = -x + xz$ $\dot{z} = -0.2z + x^2$	$(0, 0, 0)$ $(\pm 0.4472, \mp 0.4472, -1)$	$-1.6180, -0.2, 0.6180$ $-1.2863, \mathbf{0.0431} \pm 0.5560i$
Family-f $f = -1$ Lorenz-like attractor	$\dot{x} = -x + y$ $\dot{y} = x - y - xz$ $\dot{z} = -0.4z + x^2 - 1$	$(-2.5, 0, 0)$ $(\pm 1, \pm 1, 0)$	$-2.8708, -0.4000, 0.8708$ $-2.4121, \mathbf{0.0060} \pm 0.9106i$
Family-f $f = 0.2$ Chen-like attractor	$\dot{x} = -x + y$ $\dot{y} = x + 0.2y - xz$ $\dot{z} = -0.4z + x^2 - 1$	$(-2.5, 0, 0)$ $\left(\pm\frac{\sqrt{35}}{5}, \pm\frac{\sqrt{35}}{5}, -1\right)$	$-2.3647, -0.4000, 1.5647$ $-1.9387, \mathbf{0.2693} \pm 1.1712i$
Family-g $g = -0.4$ Chen-like attractor	$\dot{x} = -x + y$ $\dot{y} = x - 0.4y - xz$ $\dot{z} = -0.1z + x^2 - 1$	$(0, 0, -10)$ $(\pm 1.0296, \pm 1.0296, 0.6)$	$-4.0302, -0.1000, 2.6302$ $-1.9734, \mathbf{0.2367} \pm 1.0091i$
Family-g $g = 0.5$ Super Chen-like attractor	$\dot{x} = -x + y$ $\dot{y} = x + 0.5y - xz$ $\dot{z} = -0.1z + x^2 - 1$	$(0, 0, -10)$ $(\pm 1.0724, \pm 1.0724, 1.5)$	$-3.6504, -0.1000, 3.1504$ $-1.5388, \mathbf{0.4694} \pm 1.1289i$

Table 2. Equilibria and Jacobian eigenvalues of several Lorenz-like and Chen-like systems (*Continued*).

Systems	Equations	Equilibria	Eigenvalues
Family-h	$\dot{x} = -x - y$	$(0, 0, -1.2500)$	$-1.8255, -0.8000, 1.7255$
$h = 0.8$	$\dot{y} = -x + 0.9y + xz$		
Lorenz-like attractor	$\dot{z} = -0.8z + y^2 - 1$	$(\pm 1.5874, \mp 1.5874, -1.9000)$	$-0.9712, \mathbf{0.0356} \pm 2.2777i$
Family-h	$\dot{x} = -x - y$	$(0, 0, -2.2222)$	$-2.0809, -0.4500, 1.9809$
$h = 0.45$	$\dot{y} = -x + 0.9y + xz$		
Chen-like attractor	$\dot{z} = -0.45z + y^2 - 1$	$(\pm 1.3620, \mp 1.3620, -1.9000)$	$-0.9090, \mathbf{0.1795} \pm 2.0122i$
Sprott C	$\dot{x} = -x + y$		
$i = 1$	$\dot{y} = -xz$	$(\pm 1, \pm 1, 0)$	$-1, \pm 1.4142i$
Chen-like attractor	$\dot{z} = y^2 - 1$		
Sprott C	$\dot{x} = -x + y$		
$i = 0.8$	$\dot{y} = -xz$	$\left(\pm \frac{2\sqrt{5}}{5}, \pm \frac{2\sqrt{5}}{5}, 0\right)$	$-1, \pm 1.2650i$
Chen-like attractor	$\dot{z} = y^2 - 0.8$		
Family-j	$\dot{x} = -0.4x + y$	$(0, 0, -10)$	$-3.3850, -0.1000, 3.2850$
$j = 0.1$	$\dot{y} = x + 0.3y - xz$		
Strange Chen-like attractor	$\dot{z} = -jz + y^2 - 1$	$(\pm 2.6363, \pm 1.0545, 1.1200)$	$-0.3939, \mathbf{0.0969} \pm 2.3742i$
Family-j	$\dot{x} = -0.4x + y$	$(0, 0, -5)$	$-2.5243, -0.20000, 2.4243$
$j = 0.2$	$\dot{y} = x + 0.3y - xz$		
Strange Chen-like attractor	$\dot{z} = -jz + y^2 - 1$	$(\pm 2.7658, \pm 1.1063, 1.1200)$	$-0.3962, \mathbf{0.0481} \pm 2.4851i$
Family-k	$\dot{x} = -x + y$	$(0, 0, -1.572)$	$-0.5000, -0.4904, 0.9624$
$k = 0.786$	$\dot{y} = -x - 0.1y - yz$		
Asymmetrical attractor	$\dot{z} = -0.5z + x^2 - 0.786$	$(\pm 0.4858, \pm 0.4858, -1.1000)$	$-0.9857, \mathbf{0.2429} \pm 0.6479i$
Family-k	$\dot{x} = -x + y$	$(0, 0, -1.62)$	$-0.5066, -0.5000, 1.0266$
$k = 0.81$	$\dot{y} = -x - 0.1y - yz$		
Asymmetrical attractor	$\dot{z} = -0.5z + x^2 - 0.81$	$(\pm 0.5099, \pm 0.5099, -1.1000)$	$-1.0099, \mathbf{0.2549} \pm 0.6708i$